PIPE LINE EFFECTS ON THE DYNAMIC RESPONSE OF THE LIQUID

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Certified that this work has been carried out under my supervision and that the work has not been submitted elsewhere for a degree.

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SYNOPSIS

Pipe Line Effects On The Dynamic Response of the Liquid is a thesis submitted in partial fulfilment of the requirements for the degree of M. Tech., by C. Padmanabha to the Department of Mechanical Engineering, Indian Institute of Technology, Kanpur in the month of May, 1969.

This thesis analyses the effect of the longitudinal and transverse waves along a soft, elastic tube on the frequency response of pipe line flow.

The exciting force for the longitudinal waves along the tube is assumed to be given by the shear drag oscillations on the walls of the tube. The simplified tube momentum equation is solved for the above exciting force by considering the tube as rigidly fixed at both ends. The exciting force is included in terms of Dirac delta function, the magnitude of which is determined by the exciting force at any point considered. Then the total effect of the exciting forces at all points are included by integrating w.r.t.x from O to L. After obtaining the longitudinal amplitude and the velocity of the tube, fluid momentum equation in the axial direction is solved for the velocity perturbations by considering the velocity of the tube as a boundary condition at the inner radius of the tube. Perturbations in velocity and pressure response are added to the main velocity and pressure response, to obtain the total dynamic response of the fluid flow.

The exciting force for the transverse waves along the tube is assumed to be given by the force due to the pressure fluctuations on the walls of the tube. The amplitude and the velocity of the tube in the transverse direction is obtained for the above exciting force as explained for the case of longitudinal waves along the tube. The simplified momentum equation in the radial direction is solved for the radial perturbations velocity by considering the radial velocity of the tube as a boundary condition at the inner radius of the tube. Then the simplified continuity equation is solved to obtain the axial velocity perturbations by assuming zero velocity perturbation at the input end of the pipe line. With this as a boundary condition along the axis of the tube, the axial fluid momentum equation is solved to obtain axial velocity perturbations.Total response of the fluid is obtained by adding the main response to the perturbations response.

The results show that the effect of the longitudinal wave along the tube is very small as compared to effect of the transverse waves along the tube on the dynamic response of the fluid for a particular value of pipe line damping. The effect of the tu seemed to be increased with the increase of viscosity of the liqu

LIST OF SYMBOLS

A_c	area of cross section of the tube material, L^2 *
A_p	pipe line area, L^2
A_L	area of the mercury column, L^2
a_n	the series first used in Equation (4.41)
B	effective bulk modulus of the liquid, M/LT^2
B'	bulk modulus of the liquid, M/LT^2
b	$b(x, j\omega)$, the function first used in Equation (5.26)
b_n	the series first used in Equation (4.20)
b_n	the series first used in Equation (5.16)
C_1	constant first used in Equation (4.16)
C_2	constant first used in Equation (4.20)
C_3	constant first used in Equation (5.14)
C_4	constant first used in Equation (5.16)
C_5	constant first used in Equation (5.24)
C_6	constant first used in Equation (5.37)
C_7	defined in page 35
C_8	defined in page 37
C_9, C_{10}	defined in page 52
C_{11}	constant first used in Equation (4.34)
C_{22}, C_{33}	constant first used in Equation (5.37)
C_{nx1}, C_{nx2}	first used in Equation (4.42)
C_{nx3}, C_{nx4}	first used in Equation (4.45)
C_{nx5}, C_{nx6}	first used in Equation (5.44)
C_{nx7}, C_{nx8}	first used in Equation (5.47)

* The three symbols L, M and T used after the description of the symbols denote the dimensions of the length, the mass and the time respectively.

C_{x1}, C_{x2} defined in page 34

C_{x3}, C_{x4} defined in page 37

C_{n1}, C_{n2} defined in page 51 and 52

c_o isentropic phase velocity, L/T

c_n the series first used in Equation (5.43)

d_n the series first used in Equation (5.43)

E modulus of elasticity M/LT^2

f frequency cycles/sec, 1/T, used in Figures

G shear modulus, M/LT^2

G_1 constant first used in Equation (3.20)

G_3 constant first used in Equation (4.5)

G_4 constant first used in Equation (4.33)

G_5 constant first used in Equation (4.35)

G_6 constant first used in Equation (5.33)

G_n the series first used in Equation (4.17)

g $g(x, j\omega)$, the function first used in Equation (4.24)

H_1 constant first used in Equation (3.20)

H_3 constant first used in Equation (4.5)

H_4 constant first used in Equation (4.33)

H_5 constant first used in Equation (4.35)

H_6 constant first used in Equation (5.33)

H_7 constant first used in Equation (5.37)

H_n the series first used in Equation (5.13)

h $h(x, j\omega)$, the function first used in Equation (3.8)

I_o modified Bessel's function of the zero order

I_1 modified Bessel's function of the first order

j	imaginary quantity, $\sqrt{-1}$
K	complex quantity, first used in Equation (3.8)
K_S	spring constant per unit area, M/L^2T^2
K_2	constant first used in Equation (3.15)
L	length of the pipe line, L
m_L	mass of the mercury column, M
m_1	constant used in Equation (C.1)
n	the polytropic index
P	the amplitude of pressure including the effect of longitudinal waves in the tube, M/LT^2
$\overline{P}$	the amplitude of pressure including the effect of transverse waves in the tube, M/LT^2
P_o	the amplitude of the input pressure, M/LT^2
P'_o	the amplitude of the input pressure perturbations, M/LT^2
P_L	the amplitude of the output pressure, M/LT^2
P'_L	the amplitude of the output pressure perturbations, M/LT^2
P_x	the amplitude of pressure at any axial position, M/LT^2
p	the instantaneous pressure, M/LT^2
$\overline{p}$	the Laplace transform of p
p'_x	the amplitude of pressure perturbations at any point, M/LT^2
p'	the amplitude of pressure perturbations at any point due to the longitudinal waves in the tube, M/LT^2
R	the linearized resistance coefficient, M/L^3T
R_i	the inside radius of the tube, L
R_o	the outside radius of the tube, L
r	the radial coordinate, L
s	the Laplace operator, complex quantity

t	the time coordinate, T
U_r	the amplitude of the radial flow velocity, L/T
u	instantaneous radial flow velocity, L/T
u'	the perturbations in the instantaneous radial flow velocity, L/T
u'_r	the perturbations in the amplitude of u', L/T
V	fluid volume, L^3
V	the amplitude of the flow velocity including the effect of longitudinal waves in the tube, L/T
$\overline{V}$	the amplitude of the flow velocity including the effect of transverse waves in the tube, L/T
V_x	the amplitude of the flow velocity at any axial position, L/T
$\overline{V}_x$	the cross sectional average of V_x, L/T
V_o	the amplitude of the input velocity, L/T
V'_o	the amplitude of the input velocity perturbations, L/T
V_L	the amplitude of the output flow velocity, L/T
V'_L	the amplitude of the output flow velocity perturbations, L/T
v	the instantaneous axial flow velocity, L/T
$\overline{v}$	the Laplace transform of v.
v'_x	the amplitude of the axial flow velocity perturbations, L/T
$\overline{v}'_x$	the cross sectional average of v_x, L/T
$\overline{v}'$	the cross sectional average of the axial flow velocity perturbations at any point due to the longitudinal waves in the tube, L/T
W_L	the amplitude of the velocity of the mercury column, L/T
w_L	the instantaneous velocity of the mercury column, L/T
x	the axial coordinate, L

y	the radial dilation of the tube, L
$\bar{y}$	the Laplace transform of y
y_x	the amplitude of the radial dilation y in the tube wall at any point, L
y_{xx_o}	the amplitude of the radial dilation y in the tube wall due to the forcing function at $x = x_o$, L
yz	the instantaneous longitudinal oscillations in the tube wall, L
$\bar{z}$	the Laplace transform of z
z_x	the amplitude of z at any point, L
z_{xx_o}	the amplitude of z due to the forcing function at $x = x_o$,L
α	complex quantity used in Equation (4.16)
α_T	complex quantity used in Equation (5.12)
β,	complex quantity used in Equation (3.19)
γ_1, γ_2	complex quantity used in Equation (5.31)
ϵ	a small positive quantity
$K + \frac{4}{3}\eta$	visco-elastic damping property, M/LT
λ	Lame's constant
μ	dynamic viscosity of the liquid, M/LT
ν	kinematic viscosity of the liquid, M/LT
ρ	density of the liquid, M/L^3
ρ'	density of the tube material, M/L^3
ϕ	complex quantity used in Equation (5.4)
ψ	complex quantity used in Equation (4.4)
ω	frequency, rad/sec, 1/T

CONTENTS

CHAPTER I

INTRODUCTION

1.1 Physical Considerations

Hydraulic transmission lines have been used in many industrial, research and military fields for high performance automatic control systems. The hydraulic servomechanisms, operating at high pressure, are found more useful for high performance automatic control systems, especially in the air craft and rocket applications for the following reasons.

1. Very high power amplications are obtained with such systems.
2. Very high speed of response are possible.
3. High power to weight ratio characteristics of these systems.

The hydraulic liquids like water, oil etc. are used in measuring instruments like monometers, flow meters, transducers,etc. Dynamic measurements made by these instruments are likely to have some error due to the dynamics of the liquid, elasticity of the interconnecting tubes etc.

The effect of interconnecting lines may usually be neglected for the low speed operations and measurements. But for high performance and high speed operations the effect of interconnecting lines may be predominant 'it is particularly so in the use of elastic tube applications' such that the system performance may change significantly. The effect of interconnecting lines may be important at all frequencies, when long or moderately long pipe lines are used.

If the fluid in these pipe lines were to be inviscid and absolutely incompressible and the lines rigid, the signal transmission through the fluid medium would have been instantaneous with the whole fluid medium in the pipe line acting as a single column. But, since no real fluid is incompressible and inviscid, this is not possible in practice. Any disturbance of the fluid at any point gives rise to waves carrying these disturbances, and these waves travel in all directions. If the radius of the pipe line is small compared to the length of the tube, the effect of the disturbance travelling in the radial directions is very small since the radial waves are quickly damped aut because of high frequency radial fluid oscillations. These effects change the output response significantly as compared to the signal applied at the input end of the pipe line.

The compressibility of the liquid together with its inertia gives rise to a finite velocity for transmission of small signals, called the phase velocity or the velocity of wave, which causes certain time delay in signal transmission in addition to the dynamic behaviour of the output response. For small disturbances the phase velocity can be calculated from the linear wave equations for both undamped and damped sound waves. Knowing the phase velocity, the transient or the frequency response of liquid in the pipe line can be estimated by using the linear wave equations applicable to liquids excited by small disturbances. For large disturbances and

high pressure operations, the transient and frequency response are governed by nonlinear simultaneous partial differential equations and, hence the analysis becomes more complicated. In this case phase velocity depends not only on the compressibility and the density, but also on the pressure level, the pressure disturbance amplitude, the rate at which pressure are varied, the flow velocity and viscous friction.

With finite disturbance signals applied to the liquid in a tube, waves may be transmitted in several modes and each mode has a unique wave velocity and damping characteristics. The major mode of wave transmission is due to the interaction of the inertia of the liquid and the capacitance effect of a compressible liquid in an elastic tube. Other wave modes that have been observed are the transverse, the longitudinal and the flexural wave transmission in the elastic tubes. The transverse waves are noticeable in tubes having relatively low elastic modulus as compared to the compressible property of the fluid or in pipe lines subjected to extreme fluid pressure variations. The longitudinal waves are noticeable in tubes which are elastically soft and are induced either by the shear drag oscillations at the wall or by a sudden change in area of the pipe line at the ends. The flexural waves in pipe lines, which show oscillations in a direction perpendicular to the axis of the pipe line, depend on the type of line supports and are caused to some extent by the longitudinal oscillations at the end of a curved pipe line.

The problem of the linear analysis can be analysed by several methods. A few most useful methods of analysis are the method of separation of variables, the Laplace transformation method and steady state analysis. The Laplace transformation can be used both for the transient analysis and the steady state analysis. The separation of variables can also be used for the transient case, but it is well suited for the steady state analysis. These methods analyse the response of the fluid medium in pipe lines from the linearized equations.

There are two methods for analysing the signal attenuation and dispersion effects caused by the shear stresses at the tube walls due to the viscosity of the fluid. The first method consists of solving the continuity and momentum equations without neglecting the radial velocity gradients in the shear stress terms contained in the momentum equation. The solution for the linear damped wave equation is obtained in terms of the modified Bessel function. The second method consists of replacing the shear stress terms in the momentum equation by a term which is a product of the linearized resistance coefficient and the cross sectional average flow velocity.For a given tube and fluid, it is possible to find a relationship between the frequency and the linearized resistance coefficient using the linear theory for the oscillatory laminar fluid flow in pipe line. In addition to the above effects, damping is known to have a noticeable effect on the phase velocity and

hence on the resonant frequencies. Linear damped wave equations can be used to analyse the effect of the viscosity variations and of the linearized coefficient variations on the effective phase velocity.

In soft elastic tubes, the gradients in axial flow velocity perturbations which are caused by either radial velocities in the fluid due to the transverse oscillations of the tube walls or by longitudinal oscillations in the tube walls, introduce an additional effect on the dynamic flow of the fluid. These perturbations in the radial gradient of axial flow velocity introduce additional damping effects in the dynamic flow of fluids in tubes (Ref.14). The radial gradients in axial flow velocity and hence the damping effects can be further influenced by the flexural mode of wave transmission in the pipe line.

For the linear analysis of the problem of the dynamic response of the liquid in pipe lines, the properties of the liquid such as the compressibility and the viscosity are assumed to be constant. But these properties are known to vary considerably with temperature and pressure. In a system where the flow velocities and disturbances are resonably small, the effect of the temperature on the variation of viscosity and compressibility is not usually of much concern in the analysis of the dynamic response of liquid in a pipe line because the liquid and the line has comparatively high specific heats which give rise to very small temperature changes in the system due to negligible energy changes.

1.2 Earlier Work:

Origin of the study of sound waves in long tubes has been studied at the time of Kirchoff[1*]. Later the transmission of sound in small tubes for various end conditions was studied by Rayleigh[2]. Since then, a great deal of work has been carried out on transmission of the sound waves in tubes.

The purpose of early investigation on unsteady fluid flow in straight pipes was to analyse the surging phenomena, or water hammer, in power plants employing large diameter conduits and to determine the velocity of sound in the fluid. The classical work in this direction was carried out by Joukowsky[3] and Alleivi[4] around 1900.

Wood[5] solved the water hammer problem using a damped wave equation consisting of the linearized resistance coefficient, assuming steady Poiseuille flow. Rich[6] has given solution for damped (using linearized resistance coefficient) and undamped wave equations for several cases using the method of Laplace transformation. This method is more suitable for estimation of both the transient and the frequency response. Iberall[7] has given an interesting analysis for the behaviour of fluid in the transmission line with a pressure sensing instrument at the load end. Iberall then modified the analysis to add the effects of fluid compressibility, finite pressure excess, inertia of the fluid medium, finite length of tubing and nonlinear effects in

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* Superscript numbers refer to references.

the theoretical analysis. Nichols[8] has analysed the problem for a semi-infinite pneumatic transmission lines for frequency response assuming the properties of fluid to be linear. He made use of the term called skin effect for the analysis of the frequency dependent viscous damping effects, to estimate the attenuation and dispersion effects caused by the viscosity of the fluid. This analysis estimates the frequency dependent viscous damping effects by neglecting the compressibility effects of the transmission medium in the pneumatic lines.

Thomson[9] has given an analysis for the existence of the several modes of wave transmission in liquids in a semi-infinite tube by considering the two dimensional undamped wave equation. His experiment gives no confirmation of the existence of these waves and he shows that below the cut off frequency of the first and the second mode these waves do not exist whereas slightly above the cut off frequencies the phase velocity for these waves becomes infinite. Morgan and Kiely[10] have given a method to analyse the effect of viscosity of liquids and the internal damping in a thin walled semi-infinite elastic tube on the phase velocity. Gerlach and Parker[11] have given an analysis to investigate the symmetric modes of wave propagation for a viscous, compressible liquid in a cylindrical conduit . Boundary conditions for both rigid and elastic walls are imposed and the resulting characteristic equation is solved for the spatial attenuation factor and phase velocity for several modes.

Brown[12] has given an analysis for the dynamic response of pipe lines in terms of Laplace operator s and the modified Bessel's functions. As this solution is obtained in terms of Laplace operator, it is valid for both frequency and transient response analysis. The computed results of the above analysis used approximate relationships for the modified Bessel's function, hence the results presented for the phase velocity against the nondimensional frequency are not accurate for nondimensional frequencies smaller than 9.

D' Souza and Oldenburger[13] have given a method of estimating the frequency response of a long pipe line including the end effects. They have considered the longitudinal mode of pipe line oscillations as caused by the pipe line elasticity and the physical characteristics of the load block such as its inertia and damping. These longitudinal oscillations at the load block caused by the area difference between the area of the pipe line and the area of the load block on which fluid pressure is acting. Hence this problem is set up in a manner that holds good only if there is an area difference between the area of the pipe line and the effective area of a freely supported load system, and if the pipe line is straight.

Srinivas[14] has developed the relationship between the linearized resistance coefficient and frequency response of flow velocity. It has been shown that the two methods using the linearized resistance coefficient and the linearized viscous flow analysis for flow are the same when the frequency

variations in the linearized resistance coefficient as obtained from the developed relationship between the two are considered. Attempt has been made to include the other modes of wave transmission as caused by the longitudinal oscillations in the pipe line excited by the shear drag oscillations on the pipe wall and the transverse oscillations in the pipe line excited by pressure fluctuations in the fluid. An approximate method for the estimation of the additional resistance coefficient caused by the transverse vibrations in the pipe line has also been developed.

1.3 Object of this Investigation:

Review of the earlier work shows that much has been done on the theoretical and the experimental study of the dynamic response of the fluid medium in pipe line for a few restricted cases. But the effect of the wave propagation along the tube material on the dynamic response of the fluid has not been solved in detail. The effects of the tube for the linear case have been analysed by D'Souza and Oldenburger[13], assuming the exciting force is due to the area difference at the load end. Attempt has been made to include the effect of the wave propagation along the tube caused by shear drag oscillations and internal pressure fluctuations in the tube by Srinivas[14].

This thesis makes use of the frequency response analysis for the dynamic response of the fluid flow. The effects of the longitudinal and transverse waves are included separately by assuming that the tube is rigidly fixed at both ends. The exciting force is included in terms of Dirac delta function, the magnitude of which is determined by the exciting force at any point considered. Then the fluid momentum and continuity equations are solved for the velocity perturbations and pressure perturbations. To get the total response, the velocity perturbations and pressure perturbations are added to the fluid velocity and pressure response. The constants are evaluated by assuming suitable end conditions.

Chapter II gives the simplified fluid momentum and continuity equations. Appendix A gives the detailed study of the order of magnitude of terms in the fluid momentum and continuity equations. Chapter III gives the frequency response analysis for the dynamic flow of the fluid, as given in the previous literatures[13,14]. Chapter IV gives the analysis to include the effect of the longitudinal waves along the tube on the dynamic response of the fluid. Chapter V gives the analysis to include the effect of the transverse waves along the tube on the dynamic response of the fluid. Chapter VI gives the detailed discussion of the computed results.

CHAPTER II

ASSUMPTIONS AND BASIC EQUATIONS FOR THE FLUID FLOW

2.1 Differential Equations for the Fluid Flow:

As this analysis is dealing with the pipe flow, it is convenient to use the differential equations in cylindrical coordinates. Let x be the axis along the axis of the tube. By assuming axial symmetry of the flow the derivatives and the velocity component in the circumferential direction can be neglected. Assuming the origin lies on the axis of the tube

continuity equation is given by

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial x} = 0 \qquad (2.1)$$

Equation of motion is given by[15]

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right] \qquad (2.2)$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial x^2}\right] \qquad (2.3)$$

The isentropic equation for the liquid is assumed to be of the form

$$\frac{\partial p}{\partial \rho} = \frac{B}{\rho} \qquad (2.4)$$

If the pipe line is very long compared to the inside diameter and if the pipe line is quite rigid, the radial velocity and its gradients and the pressure in the radial

direction are damped out quickly. Hence, the radial velocity and the momentum equation in the radial direction can be neglected. A detailed discussion of the order of magnitude of terms is given in Appendix A. Simplified form of the continuity and momentum equations are written as

$$\frac{1}{\rho}\ \frac{\partial \rho}{\partial t} + \frac{\partial \bar{v}}{\partial x} = 0 \tag{2.5}$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\ \frac{\partial p}{\partial x} - \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\ \frac{\partial v}{\partial r} \right] = 0 \tag{2.6}$$

Combining Equations (2.4) and (2.5) yields the following equation

$$\frac{1}{B}\ \frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{v}}{\partial x} = 0 \tag{2.7}$$

Replacing the shear stress term in Equation (2.6) by an equivalent linearized resistance coefficient R, Equation (2.6) reduces to the form

$$\frac{\partial \bar{v}}{\partial t} + \frac{1}{\rho}\ \frac{\partial \bar{p}}{\partial x} + \frac{R\bar{v}}{\rho} = 0 \tag{2.8}$$

The above equation can be used to analyse the problem for the linearized resistance coefficient approach.

The isentropic wave propagation velocity in a compressible fluid for small perturbations is given by the equation

$$c_o = \sqrt{dp/d\rho}$$

From the equation of state of liquid,

$$dp/d\rho = B/\rho = c_o^{\,2} \tag{2.9}$$

2.2 Effective Bulk Modulus:

Due to the law compressibility of the liquid, the elastic property of the pipe line will have some influence on the effective bulk modulus which in turn affects the overall response of the system. The elasticity of the tube material causes a reduction in effective bulk modulus which reduces the wave propagation of the liquid. A method of estimating the effective bulk modulus was given by Srinivas[14]. The method estimates the effective bulk modulus of liquid in thick walled pipe line when the variations in the fluid tube dilation are small as compared to its radius. The details of the above method is given in Appendix C, and the effective bulk modulus is approximated by the equation

$$\frac{1}{B} = \frac{1}{B'} + \frac{2R_i^2 \left[(1-\sigma) + (1+\sigma) R_o^2/R_i^2\right]}{E\,(R_o^2 - R_i^2)} \qquad (2.10)$$

2.3 Assumptions:

The following assumptions are made for the dynamic flow of viscous liquid in a pipe line in order to facilitate a linear analysis of the general wave equations and the equation of motion.

1. Fluid flow is axially symmetrical and the end effects are relatively small for a long pipe line.
2. The flow is laminar with constant fluid viscosity and bulk modulus.
3. The body forces are negligible.
4. The fluid compressibility is relatively small compared to the fluid oscillations.

CHAPTER III

ANALYSIS FOR THE LINEAR CASE

3.1 Frequency Response Analysis:

From Equations (2.5) and (2.6) the simplified continuity and momentum equation for the linear case is given by

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} = 0 \qquad (3.1)$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right] = 0 \qquad (3.2)$$

By combining Equations (2.4) and (3.1) the following equation is obtained.

$$\frac{1}{B} \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} = 0 \qquad (3.3)$$

The separation of variable method uses the substitution $j\omega$ for d/dt to separate the time derivative and make it suitable for steady state response analysis. The above substitution reduces the partial differential equation into an ordinary differential equation called the Helmoltz equation.

It is assumed that the solution of Equations (3.2) and (3.3) be of the form

$$p(x, t) = P_x \, e^{j\omega t} \qquad (3.4a)$$

$$v(x,r,t) = V_x(r) \, e^{j\omega t} \qquad (3.4b)$$

$$u(x,r,t) = U_x(r) \, e^{j\omega t} \qquad (3.4c)$$

Substituting Equation (3.4) in Equation (3.2), Equation (3.2) reduces to the form

$$\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}\frac{\partial V_x}{\partial r} - \frac{j\omega}{\nu} V_x - \frac{1}{\mu}\frac{dP_x}{dx} = 0 \qquad (3.5)$$

Substituting $T = V_x + \frac{1}{j\omega\rho}\frac{dP_x}{dx}$ (3.6)

Equation (3.5) reduces to the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{j\omega}{\nu} T = 0 \qquad (3.7)$$

The above equation has only one solution which is finite at $r = 0$, and this is given by

$$T = h(x, j\omega)\ I_o(Kr) \qquad (3.8)$$

where I_o is the modified Bessel's function of order zero and

$$K = \sqrt{j\omega/\nu}$$

Combining Equations (3.6) and (3.8), the following equation is obtained.

$$V_x = h(x, j\omega)\ I_o\ (Kr) - \frac{1}{j\omega\rho}\frac{dP_x}{dx} \qquad (3.9)$$

Boundary conditions are

at $r = R_i$ $\quad V_x(r = R_i, x, j\omega) = 0$ (3.10a)

$r = 0$ $\quad \partial V_x/\partial r = 0$ (3.10b)

Using the first boundary condition in Equation (3.9) yields

$$\frac{d\ P_x}{dx} = j\omega\rho\ \ h(x, j\omega)\ I_o(K\ R_i) \qquad (3.11)$$

Now combining Equations (3.11) and (3.9) yield:

$$V_x = h(x, j\omega) \left[I_o (Kr) - I_o(K R_i) \right] \qquad (3.12)$$

Equation (3.12) shows that the amplitude of axial flow velocity has a maximum value at the centre and decreases to zero at the walls of the tube. The average cross sectional axial flow velocity and pressure can be defined by the following equations.

$$\overline{V}_x = \frac{2\pi}{\pi R_i^2} \int_0^{R_i} V_x \, rdr \qquad (3.13)$$

$$\overline{P}_x = \frac{2\pi}{\pi R_i^2} \int_0^{R_i} P_x \, r \, dr \qquad (3.14)$$

From Equations (3.12) and (3.13)

$$\overline{V}_x = K_2 \; h(x, j\omega) \qquad (3.15)$$

where $K_2 = \dfrac{2}{K R_i} \; I_1 (K R_i) - I_o (KR_i)$

As it is assumed that the pressure is a function of x only, the average cross sectional pressure can be written as

$$\overline{P}_x = P_x \qquad (3.16)$$

Substitution of Equation (3.4) in Equation (3.3) yields,

$$\frac{d \overline{V}_x}{dx} = - \frac{j\omega}{B} P_x \qquad (3.17)$$

The average axial flow velocity is used in this equation

because of the assumption of axial symmetry and negligible radial velocity. Differentiation of Equation (3.17) yields

$$\frac{dP_x}{dx} = \frac{jB}{\omega} \frac{d^2\overline{V}_x}{dx^2} \qquad (3.18)$$

Substituting Equations (3.11) and (3.15) in Equation (3.18) the following equation is obtained.

$$\frac{d^2\, h(x,\, j\omega)}{dx^2} - \gamma^2 \; h(x,\, j\omega) = 0 \qquad (3.19)$$

Where $\gamma^2 = \dfrac{\beta^2\omega^2}{c_o^{\,2}}$; $\beta^2 = \dfrac{I_o(K\ R_i)}{K_2}$

Solving Equation (3.19) for $h(x,\, j\omega)$ yields the solution

$$h(x,\, j\omega) = G_1 \cosh(\gamma\, x) + H_1 \sinh(\gamma\, x) \qquad (3.20)$$

Substituting Equation (3.20) in Equation (3.15) the average cross sectional axial velocity component is given by

$$\overline{V}_x = K_2 \left[G_1 \cosh(\gamma\, x) + H_1 \sinh(\gamma\, x) \right] \qquad (3.21)$$

From Equation (3.17) pressure amplitude is given by

$$P_x = \frac{jB}{\omega} \frac{d\overline{V}_x}{dx} \qquad (3.22)$$

Substitution of Equation (3.21) in Equation (3.22) the pressure component is given by

$$P_x = \frac{j\rho\, c_o\, I_o\, (KR_i)}{\beta} \left[G_1 \sinh(\gamma\, x) + H_1 \cosh(\gamma\, x) \right] \qquad (3.23)$$

By neglecting the effect of tube and using Equations

(3.21) and (3.23), and using the following boundary conditions

$$P_x\ (\ x = 0,\ j\omega\ ) \ = \ P_o$$
$$\overline{V}_x\ (\ x = 0,\ j\omega\ ) \ = \ V_o$$
$$P_x\ (\ x = L,\ j\omega\ ) \ = \ P_L$$
$$\overline{V}_x\ (\ x = L,\ j\omega\ ) \ = \ V_L \qquad (3.24)$$

the constants G_1 and H_1 are obtained as

$$G_1 = \frac{V_o}{K_2} \quad ; \quad H_1 = - \frac{j\ \beta\ P_o}{\rho\ c_o\ I_o\ (K\ R_i)} \qquad (3.25)$$

Then from Equations (3.21), (3.23), (3.24) and (3.25) the following equations for the pressure and the velocity response are obtained.

$$\frac{P_L}{P_o} = \cosh(\gamma\ L) - \frac{V_o}{P_o}\ \frac{\sinh(\gamma\ L)}{D} \qquad (3.26)$$

$$\frac{V_L}{V_o} = \cosh\ (\gamma\ L) - \frac{P_o}{V_o}\ D\ \sinh(\gamma\ L) \qquad (3.27)$$

where $D = \dfrac{j}{\beta\ \rho\ c_o}$

Dividing Equation (3.27) by Equation (3.26) and re-arranging the terms yields the following equation.

$$\frac{P_o}{V_o} = \frac{P_L\ \cosh(\gamma\ L) + V_L/D\ \sinh(\gamma\ L)}{P_L\ D\ \sinh(\gamma\ L) + V_L\ \cosh(\gamma\ L)} \qquad (3.28)$$

By substituting the above equation in Equations (3.26) and (3.27) and by further simplification, the frequency response equations for the pressure and the velocity are obtained as

$$\frac{P_L}{P_o} = \frac{1}{\cosh(\Gamma L) - j \beta \rho c_o \;(V_L/P_L) \sinh(\Gamma L)} \qquad (3.29)$$

$$\frac{V_L}{V_o} = \frac{1}{\cosh(\Gamma L) + \frac{j}{\beta \rho c_o} \frac{P_L}{V_L} \sinh(\Gamma L)} \qquad (3.30)$$

Where P_L/V_L is the load impedence which depends upon the condition at the load end of the pipe line. A method of estimating the load impedence for a load system with mercury column oscillating in a vertical U-tube is given in Appendix E.

3.2 The Effect of Tube on the Dynamic Response of the Fluid:

In addition to the change in effective bulk modulus of the liquid, the wave transmission along the material of the pipe line influence the dynamic response of the fluid media. The wave transmission along the pipe line usually takes place in several modes. In the following chapters only the axial symmetrical modes are considered for the analysis. They are the longitudinal and the transverse modes of the pipe line vibrations.

CHAPTER IV

THE EFFECT OF THE LONGITUDINAL WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID

The Effect of the longitudinal waves in the tube on the Dynamic response of the liquid:

The tube momentum equation in the axial direction is given by[17]

$$\rho' \frac{\partial^2 z}{\partial t^2} = (\lambda + 2G) \frac{\partial^2 z}{\partial x^2} + (\lambda + G)\left(\frac{\partial^2 y}{\partial x \partial r} + \frac{1}{r}\frac{\partial y}{\partial x}\right) +$$

$$G\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right) + \text{damping} \qquad (4.1)$$

where z is the longitudinal oscillation amplitude

y is the transverse oscillation amplitude

The above equation is obtained by considering the compressibility of the tube material. The detailed study of Equation (4.1) is given in Appendix A for the frequency response analysis. The resulting wave equation is linearized by neglecting the terms of smaller order of magnitude and the damping effect is included by adding visco-elastic damping.[18] The resulting linearized damped wave equation is given by

$$\rho' \frac{\partial^2 z}{\partial t^2} = (\lambda + 2G) \frac{\partial^2 z}{\partial x^2} + (K + \tfrac{4}{3}\eta) \frac{\partial^3 z}{\partial t \partial x^2} \qquad (4.2)$$

For the frequency response analysis the solution of the equation is assumed to be of the form

$$z(x,t) = z_x \, e^{j\omega t} \qquad (4.3)$$

Substituting Equation (4.3) in Equation (4.2) yields the following equation.

$$\frac{d^2 z_x}{dx^2} + \psi^2 z_x = 0 \tag{4.4}$$

where $\psi^2 = \frac{\rho' \, {}^2}{\lambda + 2G + j\omega E_i}$; $E_i = K + \frac{4}{3}\eta$

General solution of Equation (4.4) without the forcing function is given by

$$z_x = G_3 \cos\psi x + H_3 \sin\psi x \tag{4.5}$$

Assuming the tube is fixed rigidly at both ends, the end conditions are

at $\quad x = 0 \qquad z_x = 0 \qquad (4.6a)$

$\quad x = L \qquad z_x = 0 \qquad (4.6b)$

Substituting (4.6) in Equation (4.5) yields

$$G_3 = 0 \text{ and } H_3 \sin\psi L = 0 \tag{4.7}$$

For nontrivial solution $H_3 \neq 0$

Hence $\sin\psi L = 0$

or $\quad \psi = \frac{n\pi}{L}$ where $n = 1, 2, \ldots \qquad (4.8)$

Substituting Equations (4.7) and (4.8) in Equation (4.5), Equation (4.5) reduces to the form

$$z_x = \Sigma \, G_n \sin\left(\frac{n\pi x}{L}\right) \tag{4.9}$$

Now considering only the excitations caused by the shear drag oscillations, the distribution of the shear forces on the tube wall can be estimated as given below.

The axial flow velocity distribution for viscous flow is given by Equation (3.12) as

$$V_x = h(x, j\omega) \left[I_o(Kr) - I_o(KR_i) \right] \tag{4.10}$$

The shear stress on the tube wall is given by the equation

$$\mu \frac{\partial V_x}{\partial r} \bigg|_{r=R_i} = \mu \, h(x, j\omega) \, K \, I_1(KR_i) \tag{4.11}$$

Let dx be an elemental length of the tube at any point x_o. Neglecting the forces acting at all the points except at $x = x_o$, the forcing function is given by

$$\text{Forcing function} = \delta(x - x_o) \, dx \, 2\pi R_i \mu \frac{\partial V_x}{\partial r} \bigg|_{r=R_i} \tag{4.12}$$

Where $\delta(x-x_o)$ is Dirac delta function having the property

$$\delta(x - x_o) = 1 \qquad \text{when } x = x_o$$

$$\delta(x - x_o) = 0 \qquad \text{when } x \neq x_o$$

Now Equation (4.2) is multiplied by the area of cross section of the tube material and elemental length dx. Then this equation is equated to the forcing function (4.12). After rearranging the resulting equation the following equation is obtained.

$$\rho' \frac{\partial^2 z}{\partial t^2} - (\lambda + 2G) \frac{\partial^2 z}{\partial x^2} - (K + \tfrac{4}{3}\eta) \frac{\partial^3 z}{\partial t \partial x^2} = \frac{\delta(x-x_o) 2\pi R_i \mu \frac{\partial v}{\partial r} \big|_{r=R_i}}{A_c} \tag{4.13}$$

where A_c = area of cross section of the tube material

Taking Laplace transform of Equation (4.13) yields

$$\rho' s^2 \bar{z} - (\lambda + 2G)\frac{d^2\bar{z}}{dx^2} - (K + \tfrac{4}{3}\eta)\, s \frac{d^2\bar{z}}{dx^2} = \frac{\mu\, \delta(x - x_o)\, 2\pi R_i \left.\frac{\partial \bar{v}}{\partial r}\right|_{r=R_i}}{A_c} \qquad (4.14)$$

By substituting $s = j\omega$ for the steady state response, Equation (4.14) reduces to the form

$$-\rho'\omega^2 z_x - (\lambda + 2G)\frac{d^2 z_x}{dx^2} - (K + \tfrac{4}{3}\eta)\, j\omega \frac{d^2 z_x}{dx^2} = \frac{\delta(x - x_o)\, 2\pi R_i \mu \left.\frac{\partial V_x}{\partial r}\right|_{r=R_i}}{A_c} \qquad (4.15)$$

Now substituting Equations (4.9) and (4.12) in Equation (4.15), and rearranging the resulting equation yield

$$\Sigma\, G_n \left(-\rho'\omega^2 + \frac{\alpha^2 n^2 \pi^2}{L^2}\right) \sin\frac{n\pi x}{L} = C_1\, h(x, j\omega)\, \delta(x - x_o) \qquad (4.16)$$

where $\alpha^2 = (\lambda + 2G) + j\omega(K + \frac{4}{3}\eta)$

$$C_1 = \frac{2\pi\, R_i \mu K\, I_1\, (KR_i)}{A_c}$$

Now by the theory of Fourier series and by the property

of the Dirac delta function

$$\int_0^L f(x)\ \delta(x - x_o)\ dx = f(x_o)$$

G_n is obtained from Equation (4.16) as

$$G_n = \frac{2C_1\ h\ (x_o,\ j\omega)\ \sin\ (n\pi x_o/L)}{L\ (-\rho'\omega^2 + \alpha^2 n^2 \pi^2/L^2)} \tag{4.17}$$

Substituting Equation (4.17) in Equation (4.9) the longitudinal amplitude of the tube due to the forcing function at $x = x_o$ is given by

$$z_{xx_o} = \frac{2C_1\ h(x_o,\ j\omega)}{L}\ \Sigma\ \frac{\sin \frac{n\pi x_o}{L} \quad \sin \frac{n\pi x}{L}}{(-\rho'\omega^2 + \frac{\alpha^2 n^2 \pi^2}{L^2})} \tag{4.18}$$

Longitudinal oscillations of the tube due to all the forcing function acting on the tube from 0 to L is given by

$$z_x = \int_0^L z_{xx_o}\ dx_o \tag{4.19}$$

From Equations (4.18) and (4.19), z_x is given by

$$z_x = C_2\ \Sigma\ b_n\ \sin \frac{n\pi x}{L} \tag{4.20}$$

where $$C_2 = \frac{4\pi^2\ R_i \mu K\ I_1(KR_i)}{L^2\ A_c}$$

$$b_n = \frac{n \left[G_1\ (1 - (-1)^n \cosh(\Gamma L)) - H_1 (-1)^n \sinh(\Gamma L) \right]}{(\Gamma^2 + n^2\pi^2/L^2)\ (-\rho'\omega^2 + \alpha^2 n^2 \pi^2/L^2)} \tag{4.21}$$

These details are given in Appendix B. The axial velocity

perturbations can be estimated by considering unit length of the tube as given below.

Velocity amplitude of the tube $= dz_x/dt = j\omega z_x$ (4.22)

Let v'_x be the axial velocity perturbations of the fluid due to the longitudinal oscillations of the tube. Then the fluid momentum equation for axial velocity perturbations is given by Equation (3.5) as

$$\frac{\partial^2 v'_x}{\partial r^2} + \frac{1}{r}\frac{\partial v'_x}{\partial r} - \frac{j\omega v'_x}{\nu} - \frac{1}{\mu}\frac{dp'_x}{dx} = 0 \qquad (4.23)$$

where p'_x is the pressure perturbations.

By going through the steps indicated in Equations (3.6), (3.7) and (3.8) the solution of Equation (4.23) is given by

$$v'_x = g(x, j\omega)\, I_o(Kr) - \frac{1}{j\omega\rho}\frac{dp'_x}{dx} \qquad (4.24)$$

Boundary conditions are

at $\quad r = 0 \qquad \frac{\partial v'_x}{\partial r} = 0$

$r = R_i \quad v'_x = j\omega z_x$ from Equation (4.22) (4.25)

Now using the boundary condition at $r = R_i$ in Equation (4.24), the following equation is obtained.

$$\frac{dp'_x}{dx} = j\omega\rho\left[\, g(x, j\omega)\, I_o\,(KR_i) - j\omega z_x \,\right] \qquad (4.26)$$

Combining Equations (4.26) and (4.24) yields the following equation

$$v'_x = g(x, j\omega) \left[I_o(Kr) - I_o(KR_i) \right] + j z_x \qquad (4.27)$$

Average velocity perturbations is obtained by the equation

$$\bar{v}'_x = \frac{2\pi}{\pi R_i^2} \int_0^{R_i} v'_x \, r \, d \, r$$

or $$\bar{v}'_x = K_2 \; g(x, j\omega) + j\omega z_x \qquad (4.28)$$

For the velocity perturbations Equation (3.18) can be written as

$$\frac{dp'_x}{dx} = \frac{jB}{\omega} \; \frac{d^2\bar{v}'_x}{dx^2} \qquad (4.29)$$

Substituting Equations (4.26), (4.28) and (4.20) in Equation (4.29) and rearranging the terms yield the following equation.

$$\frac{d^2g(x, j\omega)}{dx^2} - \gamma^2 \, g(x, j\omega) = \frac{C_2 j\,\omega}{K_2} \Sigma \left(\frac{n^2\pi^2}{L^2} - \frac{\omega^2}{c_o^2} \right) b_n \sin\frac{n\pi x}{L}$$

$$(4.30)$$

These details are given in Appendix B.

Particular integral of the above nonhomogeneous ordinary differential equation is given by

$$g_p = - \frac{C_2 \; j\omega}{K_2} \Sigma \; \frac{\left(\frac{n^2\pi^2}{L^2} - \omega^2/c_o^2\right)}{(\gamma^2 + n^2\pi^2/L^2)} \; b_n \sin \frac{n\pi x}{L} \qquad (4.31)$$

Hence the general solution of the differential equation (4.30) is given by

$$g(x, j\omega) = G_4 \cosh(\gamma x) + H_4 \sinh(\gamma x) + g_p \qquad (4.32)$$

Substituting Equations (4.20) and (4.32) in Equation (4.28) the cross sectional average velocity perturbations is given by

$$\bar{v}'_x = G_4\, K_2 \cosh(\gamma x) + H_4\, K_2 \sinh(\gamma x) + C_2\, j\omega\left(\gamma^2 + \frac{\omega^2}{c_o^2}\right) \Sigma \frac{b_n \sin \frac{n\pi x}{L}}{\left(\gamma^2 + \frac{n^2\pi^2}{L^2}\right)} \qquad (4.33)$$

Because of the reason that a unit length of the tube is considered in Equation (4.22) the average axial velocity perturbations at any point x is given by

$$\bar{v}' = \int_0^x \bar{v}'_x \, dx$$

$$\bar{v}' = G_5 \sinh(\gamma x) + H_5 \left(\cosh(\gamma x) - 1\right) + C_{11} \Sigma \frac{(1-\cos n\pi x/L)}{n\left(\gamma^2 + \frac{n^2\pi^2}{L^2}\right)} b_n \qquad (4.34)$$

where $C_{11} = \dfrac{C_2\,(\gamma^2 + \omega^2/c_o^2) L\, j\,\omega}{\pi}$

From Equation (3.17) pressure perturbations at any point x is given by

$$p' = \frac{jB}{\omega} \frac{d\bar{v}'}{dx}$$

Substitution of Equation (4.34) in the above equation yields the following equation.

$$p' = \frac{jB\gamma}{\omega} G_5 \cosh \gamma x + \frac{jB\gamma}{\omega} H_5 \sinh \gamma x + \frac{C_{11}\ jB\pi}{\omega L} \Sigma \frac{b_n \sin n\pi x/L}{(\gamma^2 + n^2\pi^2/L^2)} \tag{4.35}$$

To solve for the two constants G_5 and H_5, following assumptions are made

at $\quad x = L \quad \frac{dp'}{dx} = 0 \quad$ or $\quad \frac{d^2\bar{v}'}{dx^2} = 0$

As there is no further effect of the tube at the end of the tube, the assumption that there is no further change in pressure perturbations. is justified

at $\quad x = L \quad p' = P'_L$

$$\bar{v}' = V'_L$$

In addition to the above assumptions it is assumed that

$$\frac{P_L + P'_L}{V_L + V'_L} = \frac{P_L}{V_L} = \frac{P'_L}{V'_L}$$

By substituting the above end conditions in Equations (4.34) and (4.35) the following equations are obtained

$$G_5\, \gamma^2 \sinh \gamma L + H_5\, \gamma^2 \cosh \gamma L = - \frac{C_{11}\pi^2}{L^2} \Sigma \frac{(-1)^n\, n\, b_n}{(\gamma^2 + \frac{n^2\pi^2}{L^2})} \tag{4.36}$$

$$G_5\left[\frac{V_L'}{P_L'}\frac{jB\gamma}{\omega}\cosh(\gamma L)-\sinh(\gamma L)\right]+H_5\left[\frac{V_L'}{P_L'}\frac{jB\gamma}{\omega}\sinh(\gamma L)-\cosh(\gamma L)+1\right]=2C_{11}\sum_{1,3\cdots}\frac{b_n}{n(\gamma^2+n^2\pi^2/L^2)} \qquad (4.37)$$

The above two equations can be written in the matrix form as

$$\begin{pmatrix} \gamma^2\sinh(\gamma L) & \gamma^2\cosh(\gamma L) \\ \frac{V_L'}{P_L'}\frac{jB\gamma}{\omega}\cosh(\gamma L)-\sinh(\gamma L) & \frac{V_L'}{P_L'}\frac{jB\gamma}{\omega}\sinh(\gamma L)-\cosh(\gamma L)+1 \end{pmatrix}$$

$$\begin{pmatrix} G_5 \\ H_5 \end{pmatrix} = \begin{pmatrix} -\frac{C_{11}\pi^2}{L^2}\sum\frac{(-1)^n\, n\, b_n}{(\gamma^2+n^2\pi^2/L^2)} \\ 2C_{11}\sum_{1,3\cdots}\frac{b_n}{n(\gamma^2+n^2\pi^2/L^2)} \end{pmatrix}$$

By applying the Cramer's rule G_5 and H_5 are obtained from the above matrix as

$$G_5=\left[2C_{11}\gamma^2\cosh(\gamma L)\sum_{1,3\cdots}\frac{b_n}{n(\gamma^2+n^2\pi^2/L^2)}+\frac{C_{11}\pi^2}{L^2}\left(\frac{V'_L}{P'_L}\frac{jB\gamma}{\omega}\sinh(\gamma L)-\cosh(\gamma L)+1\right)\sum\frac{(-1)^n\, n\, b_n}{(\gamma^2+n^2\pi^2/L^2)}\right]\Bigg/\left[\frac{jB\gamma^3}{\omega}\frac{V_L'}{P_L'}-\gamma^2\sinh(\gamma L)\right]$$

(4.38)

$$H_5 = \Big[-2C_{11}\gamma^2 \sinh(\gamma L) \sum_{1,3,\ldots} \frac{b_n}{n(\gamma^2+n^2\pi^2/L^2)} -$$

$$\frac{C_{11}\pi^2}{L^2} \left(\frac{V_L'}{P_L'} \frac{jB\gamma}{\omega} \cosh(\gamma L) - \sinh(\gamma L)\right)$$

$$* \sum \frac{(-1)^n n\, b_n}{(\gamma^2 + n^2\pi^2/L^2)} \Big] \Big/ \Big[\frac{jB\gamma^3}{\omega} \frac{V_L'}{P_L'} - \gamma^2 \sinh(\gamma L) \Big]$$

(4.39)

The frequency response of the velocity including the effect of the longitudinal waves along the pipe line can be obtained by adding Equations (3.22) and (4.34).

$$V = \overline{V}_x + \overline{v}' \tag{4.40}$$

By substituting the constants G_5 and H_5 in Equation (4.34) and then substituting Equations (4.34) and (3.22) in Equation (4.40) yield

$$V = G_1 \Big[\Big[\frac{C_{11}\pi^2}{L^2} \sinh(\gamma x) \left(\frac{V_L'}{P_L'} \frac{jB\gamma}{\omega} \sinh(\gamma L) - \cosh(\gamma L) + 1\right) S_1$$

$$+ 2\, C_{11}\gamma^2 \cosh(\gamma L) \sinh(\gamma x)\, S_2 - 2C_{11}\gamma^2 \sinh(\gamma L) *$$

$$(\cosh(\gamma x) - 1)\, S_2 - \frac{C_{11}\pi^2}{L^2} (\cosh(\gamma x) - 1) \left(\frac{V_L'}{P_L'} \frac{jB\gamma}{\omega} \cosh(\gamma L)\right.$$

$$\left. - \sinh(\gamma L)\right) S_1 \Big] / C_7 + C_{11} \sum \frac{\left(1 - \cos n\pi x/L\right)}{a_n} (1 - (-1)^n *$$

$$\cosh(\gamma L)) + K_2 \cosh(\gamma x) \Big]$$

(continued on the next page....)

$$+ H_1 \Big[\Big[- \frac{C_{11}\pi^2}{L^2} \sinh(\gamma x) \left(\frac{V_L'}{P_L'} \frac{jB\gamma}{\omega} \sinh(\gamma L) - \cosh(\gamma L) + 1\right) S_3$$

$$-2\, C_{11}\gamma^2 \cosh(\gamma L) \sinh(\gamma x) S_4 + 2C_{11}\gamma^2 \sinh(\gamma L)(\cosh(\gamma x) - 1) S$$

$$+ \frac{C_{11}\pi^2}{L^2} (\cosh(\gamma x) - 1) \left(\frac{V_L'}{P_L'} \frac{jB\gamma}{\omega} \cosh(\gamma L) - \sinh(\gamma L)\right) S_3 \Big] / C_7$$

$$-C_{11} \Sigma \frac{\left(1 - \cos \frac{n\pi x}{L}\right)}{a_n} (-1)^n \sinh(\gamma L) + K_2 \sinh(\gamma x) \Big] \qquad (4.41)$$

Further simplification of Equation (4.41) yields

$$V = C_{nx1} \; G_1 + C_{nx2} \; H_1 \qquad (4.42)$$

where $C_{nx1} = \dfrac{C_{x1} \; S_1 + C_{x2} \; S_2}{C_7} + C_{11} \Sigma \dfrac{\left(1 - \cos \frac{n\pi x}{L}\right)}{a_n} \Big[1 - (-1)^n \cosh(\gamma$

$$+ K_2 \cosh(\gamma x)\Big)$$

$$C_{nx2} = \frac{-C_{x1}S_3 - C_{x2}S_4}{C_7} - C_{11} \Sigma \frac{\left(1 - \cos \frac{n\pi x}{L}\right)}{a_n} (-1)^n \sinh(\gamma L) +$$

$$K_2 \sinh(\gamma x)$$

$$C_{x1} = \frac{C_{11}\pi^2}{L^2} \Big[\frac{jB\gamma}{\omega} \frac{V_L'}{P_L'} (\sinh(\gamma L) \sinh(\gamma x) - \cosh(\gamma x) \cosh(\gamma L)$$

$$\cosh(\gamma L)) - \sinh(\gamma x) \cosh(\gamma L) + \sinh(\gamma L) \cosh(\gamma x) + \sinh(\gamma x) - \sinh$$

$$C_{x2} = 2C_{11}\gamma^2 (\cosh(\gamma L) \sinh(\gamma x) - \sinh(\gamma L) \cosh(\gamma x) + \sinh(\gamma$$

$$S_1 = \Sigma \frac{(-1)^n n^2 \left[1-(-1)^n \cosh(\Gamma L)\right]}{a_n} \quad ;$$

$$S_2 = \sum_{1,3\cdots} \frac{1-(-1)^n \cosh(\Gamma L)}{a_n} \quad ; \quad S_3 = \Sigma \frac{n^2 \sinh(\Gamma L)}{a_n} \quad ;$$

$$S_4 = \sum_{1,3\cdots} \frac{(-1)^n \sinh(\Gamma L)}{a_n}$$

$$a_n = (-\rho' \omega^2 + \alpha^2 n^2 \pi^2 / L^2)\ (\Gamma^2 + n^2 \pi^2 / L^2)^2$$

$$C_7 = \frac{jB\Gamma^3 V_L'}{\omega\ P_L'} - \Gamma^2 \sinh(\Gamma L)$$

at $\quad x = 0$

$$C_{n01} = \frac{C_{01}\ S_1 + C_{02}\ S_2}{C_7} + K_2$$

$$C_{n02} = \frac{-C_{01}\ S_3 - C_{02} S_4}{C_7}$$

$$C_{01} = 0 \quad ; \quad C_{02} = 0$$

at $\quad x = L$

$$C_{nL1} = \frac{C_{L1}\ S_1 + C_{L2}\ S_2}{C_7} + 2\ C_{11}\ S_2 + K_2 \cosh(\Gamma L)$$

$$C_{nL2} = \frac{-C_{L1}\ S_1 - C_{L2}\ S_2}{C_7} - 2C_{11}\ S_4 + K_2 \sinh(\Gamma L)$$

$$C_{L1} = \frac{C_{11}\pi^2}{L^2} \left[\frac{jB\Gamma}{\omega} \frac{V_L'}{P_L'} (\cosh(\Gamma L) - 1) \right]$$

$$C_{L2} = 2C_{11} \Gamma^2 \sinh(\Gamma L)$$

The frequency response of the pressure including the effect of the longitudinal waves along the pipe line can be obtained by adding Equations (3.23) and (4.35).

$$P = P_x + p' \tag{4.43}$$

By substituting the constants G_5 and H_5 in Equation (4.35) and then substituting Equations (4.35) and (3.23) in Equation (4.43) yield

$$\begin{aligned}
P = G_1 \Bigg[\Bigg[& \frac{C_{11}\pi^2}{L^2} \frac{jB\gamma}{\omega} \cosh(\gamma x) \left(\frac{V'_L}{P'_L} \frac{jB\gamma}{\omega} \sinh(\gamma L) - \cosh(\gamma L) + 1\right) \\
& + 2C_{11} \frac{jB\gamma}{\omega} \gamma^2 \cosh(\gamma L) \cosh(\gamma x) S_2 - 2\, C_{11} \frac{jB\gamma}{\omega} \gamma^2 \sinh(\gamma L) \cdot \\
& \sinh(\gamma x) S_2 - \frac{C_{11}\pi^2}{L^2} \frac{jB\gamma}{\omega} \sinh(\gamma x) \left(\frac{V'_L}{P'_L} \frac{jB\gamma}{\omega} \cosh(\gamma L) - \sinh(\gamma L)\right) S_1 \Bigg] / \\
& + C_{11} \frac{jB\pi}{\omega L} \Sigma \frac{n \sin(n\pi x/L)}{a_n} (1-(-1)^n \cosh(\gamma L)) + C_8 \sinh(\gamma x) \Bigg] \\
+ H_1 \Bigg[\Bigg[& - \frac{C_{11}\pi^2}{L^2} \frac{jB\gamma}{\omega} \cosh(\gamma x) \left(\frac{V'_L}{P'_L} \frac{jB\gamma}{\omega} \sinh(\gamma L) - \cosh(\gamma L) + 1\right) S_3 \\
& - 2C_{11} \frac{jB\gamma}{\omega} \gamma^2 \cosh(\gamma L) \cosh(\gamma x) S_4 + 2C_{11} \frac{jB\gamma}{\omega} \gamma^2 \sinh(\gamma L) \cdot \\
& \sinh(\gamma x)\ S_4 + \frac{C_{11}\pi^2}{L^2} \frac{jB\gamma}{\omega} \sinh(\gamma x) \left(\frac{V'_L}{P'_L} \frac{jB\gamma}{\omega} \cosh(\gamma L) \right. \\
& \left. - \sinh(\gamma L)\right) S_1 \Bigg] / C_7 - C_{11} \frac{jB\pi}{\omega L} \Sigma \frac{n \sin n\pi x/L}{a_n} 1 - (-1)^n \sinh(\gamma L)) \\
& + C_8 \cosh(\gamma x) \Bigg]
\end{aligned} \tag{4.44}$$

Further simplification of Equation (4.44) yields

$$P = C_{nx3}\ G_1 + C_{nx4}\ H_1 \tag{4.45}$$

where

$$C_{nx3} = \frac{C_{x3}\ S_1 + C_{x4}\ S_2}{C_7} + C_{11}\ \frac{jB\pi}{\omega L}$$

$$\Sigma\ \frac{n\left[\ 1-(-1)^n \cosh \Gamma L\ \right] \sin \frac{n\pi x}{L}}{a_n} + C_8\ \sinh(\Gamma x)$$

$$C_{nx4} = \frac{-C_{x3}\ S_3 - C_{x4}\ S_4}{C_7} - C_{11}\ \frac{jB\ \pi}{\omega L}\ \Sigma\ \frac{n(-1)^n \sinh(\Gamma L) \sin \frac{n\pi x}{L}}{a_n}$$

$$+\ C_8\ \cosh(\Gamma\ x)$$

$$C_{x3} = \frac{C_{11}\pi^2}{L^2}\ \frac{jB\Gamma}{\omega}\left[\frac{jB\Gamma}{\omega}\frac{V_L'}{P_L'}\ (\cosh(\Gamma x)\sinh(\Gamma L) - \sinh(\Gamma x)\cosh(\Gamma L))\right.$$

$$\left. + \cosh(\Gamma x) - \cosh(\Gamma x)\cosh(\Gamma L) + \sinh(\Gamma x)\sinh(\Gamma L)\right]$$

$$C_{x4} = \frac{2C_{11}\Gamma^2\ jB\Gamma}{\omega}\ (\cosh(\Gamma L)\cosh(\Gamma x) - \sinh(\Gamma L)\sinh(\Gamma x))$$

$$C_8 = \frac{j\rho c_o\ I_o\ (KR_i)}{\beta}$$

at $\quad x = 0$

$$C_{n03} = \frac{C_{03}\ S_1 + C_{04}\ S_2}{C_7}$$

$$C_{n04} = -\ \frac{C_{03}\ S_3 - C_{04}\ S_4}{C_7} + C_8$$

$$C_{03} = \frac{C_{11}\,\pi^2}{L^2}\frac{jB\gamma}{\omega}\left[\frac{jB\gamma}{\omega}\frac{V_L'}{P_L'}\sinh(\gamma L) + 1 - \cosh(\gamma L)\right]$$

$$C_{04} = \frac{2C_{11}\,\gamma^2\, jB\gamma}{\omega}\cosh(\gamma L)$$

at $\quad x = L$

$$X_{nL3} = \frac{C_{L3}\,S_1 + C_{L4}\,S_2}{C_7} + C_8\sinh(\gamma L)$$

$$C_{nL4} = \frac{-C_{L3}\,S_3 - C_{L4}\,S_4}{C_7} + C_8\cosh(\gamma L)$$

$$C_{L3} = \frac{C_{11}\pi^2}{L^2}\frac{jB\gamma}{\omega}(\cosh(\gamma L) - 1)$$

$$C_{L4} = \frac{2C_{11}\gamma^2\, jB\gamma}{\omega}$$

Using Equations (4.45) and (4.42) and using the boundary conditions

$$P(x = 0,\ j\omega) = P_o + P_o'$$

$$V(x = 0,\ j\omega) = V_o + V_o'$$

$$V(x = L,\ j\omega) = V_L + V'_L$$

$$P(x = L,\ j\omega) = P_L + P'_L \qquad (4.46)$$

the constants G_1 and H_1 are obtained as

$$G_1 = \frac{V_o + V_o'}{K_2} \qquad (4.47)$$

$$H_1 = \frac{P_o + P_o'}{C_{n04}} - \frac{C_{n03}}{C_{n04}}\frac{V_o + V_o'}{K_2} \qquad (4.48)$$

Substituting (4.48) in Equations (4.42) and (4.45), the following equations for the velocity and pressure response are obtained.

$$V = \frac{V_o + V_o'}{K_2} C_{nx1} + \left[\frac{P_o + P_o'}{C_{n04}} - \frac{C_{n03}}{C_{n04}} \frac{V_o + V_o'}{K_2} \right] C_{nx2} \quad (4.49)$$

$$P = \frac{V_o + V_o'}{K_2} C_{nx3} + \left[\frac{P_o + P_o'}{C_{n04}} - \frac{C_{n03}}{C_{n04}} \frac{V_o + V_o'}{K_2} \right] C_{nx4} \quad (4.50)$$

Then from Equations (4.49), (4.50) and (4.47), the following equations for the velocity and pressure response are obtained.

$$\frac{P_L + P'_L}{P_o + P'_o} = \frac{V_o + V_o'}{P_o + P_o'} \frac{C_{nL3}}{K_2} + \left[\frac{1}{C_{n04}} - \frac{V_o + V_o'}{P_o + P_o'} \frac{C_{n03}}{C_{n04} K_2} \right] C_{nL4} \quad (4.51)$$

$$\frac{V_L + V_L'}{V_o + V_o'} = \frac{C_{nL1}}{K_2} + \left[\frac{P_o + P_o'}{(V_o + V_o') C_{n04}} - \frac{C_{n03}}{C_{n04} K_2} \right] C_{nL2} \quad (4.52)$$

Dividing Equation (4.51) by Equation (4.52) and rearranging the terms yields the following equation

$$\frac{P_o+P'_o}{V_o+V'_o} = \left[\frac{V_L+V'_L}{P_L+P'_L} \left\{ \frac{C_{nL3}}{K_2} - \frac{C_{nL4}\ C_{n03}}{K_2\ C_{n04}} \right\} + \frac{C_{n03}C_{nL2}}{K_2\ C_{n04}} - \frac{C_{nL1}}{K_2} \right] \Bigg/ \left[\frac{C_{nL2}}{C_{n04}} - \frac{V_L+V'_L}{P_L+P'_L}\ \frac{C_{nL4}}{C_{n04}} \right] \quad (4.53)$$

Where $(P_L + P'_L)/(V_L + V'_L)$ is the load impedence. A method of estimating the load impedence for a load system with mercury column oscillating in a vertical U-tube is given in Appendix E. The frequency response of the system including the effect of longitudinal waves in the tube can be estimated from Equations (4.52) and (4.53) with the appropriate system parameters.

CHAPTER V

THE EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID

The Effect of the Transverse Waves in the Tube on the Dynamic Response of the Liquid:

The tube momentum equation in the radial direction is given by[18]

$$\rho' \frac{\partial^2 y}{\partial t^2} = (\lambda + 2G) \left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} - \frac{y}{r^2}\right) + (\lambda + G) \frac{\partial^2 y}{\partial r \partial x}$$
$$+ G \frac{\partial^2 y}{\partial x^2} + \text{damping} \qquad (5.1)$$

The detailed study of Equation (5.1) is given in Appendix A for the frequency response analysis. The resulting wave equation is linearized by neglecting the terms of smaller order of magnitude by selecting the solution of suitable form and the damping effect is included by adding visco-elastic damping[19]. The resulting linearized damped wave equation is given by

$$\rho' \frac{\partial^2 y}{\partial t^2} = G \frac{\partial^2 y}{\partial x^2} + \left(K + \frac{4}{3}\eta\right) \frac{\partial^3 y}{\partial t \partial y^2} \qquad (5.2)$$

For the frequency response analysis, the radial disturbances are assumed to be of the form

$$y(x,t) = y_x \, e^{j\omega t} \qquad (5.3)$$

Then Equation (5.2) is transformed to the ordinary differential equation

$$\frac{d^2 y_x}{dx^2} + \phi^2 y_x = 0 \qquad (5.4)$$

where $\phi^2 = \dfrac{\rho_1 \omega^2}{G+j\omega E_i}$: $E_i = K + \frac{4}{3}\eta$

Assuming the tube is fixed rigidly at both ends, the end conditions are given by

$$\begin{aligned} \text{at} \quad x &= 0 & y_x &= 0 \\ x &= L & y_x &= 0 \end{aligned} \tag{5.5}$$

Using these boundary conditions the solution of the differential equation (5.4) is obtained as

$$y_x = \Sigma\, H_n \sin \frac{n\pi x}{L} \tag{5.6}$$

Now considering only the excitation caused by the fluctuations in the internal pressure, the distribution of the above forces on the tube wall can be estimated as given below:

The internal pressure for viscous flow is given by Equation (3.13) as

$$P_x = C_8 \left[G_1 \sinh(\Gamma x) + H_1 \cosh(\Gamma x) \right] \tag{5.7}$$

Let dx be an elemental length of the tube at any point x_o. Neglecting the forces acting at all other points except at $x = x_o$

$$\text{Forcing function} = \delta(x-x_o)\, 2\pi\, R_i\, dx\, P_x \tag{5.8}$$

Multiplying Equation (5.2) by the area of cross section of the tube material and the elemental length dx and, equating to the forcing function (5.8) and then rearranging the terms yields

$$\rho' \frac{\partial^2 y}{\partial t^2} - G \frac{\partial^2 y}{\partial x^2} - (K + \tfrac{4}{3}\eta) \frac{\partial^3 y}{\partial t \partial x^2} = \frac{\delta(x-x_o) 2\pi R_i p}{A_c} \quad (5.9)$$

Taking Laplace transform of Equation (5.9) yields

$$\rho' s^2 \bar{y} = G \frac{d^2\bar{y}}{dx^2} - (K + \tfrac{4}{3}\eta) s \frac{d^2\bar{y}}{dx^2} = \frac{2\pi R_i \delta(x-x_o)\bar{p}}{A_c} \quad (5.10)$$

By substituting $s = j\omega$ for the steady state response the above equation reduces to the form

$$-\rho'\omega^2 y_x - G \frac{d^2 y_x}{dx^2} - (K + \tfrac{4}{3}\eta) j\omega \frac{d^2 y_x}{dx^2} = \frac{2\pi R_i \delta(x-x_o) P_x}{A_c} \quad (5.11)$$

Combining Equations (5.6) and (5.11) and rearranging, the following equation is obtained

$$\Sigma H_n(-\rho'\omega^2 + \alpha_T{}^2 n^2 \pi^2/L^2) \sin \frac{n\pi x}{L} = \frac{2\pi R_i \delta(x-x_o) P_x}{A_c} \quad (5.12)$$

where $\alpha_T{}^2 = G + j\omega(K + \tfrac{4}{3}\eta)$

Now by the theory of Fourier series and by the property of the dirac delta function H_n is obtained from Equation (5.12) as

$$H_n = \frac{4\pi R_i P_{x_o} \sin \sin n\pi x_o/L}{LA_c (-\rho'\omega^2 + \alpha_T{}^2 n^2 \pi^2/L^2)} \quad (5.13)$$

Substituting the expression for H_n in Equation (5.6) the radial amplitude of the tube due to the forcing function at any point x_o is given by

$$y_{xx_o} = C_3 \, P_{x_o} \, \Sigma \, \frac{\sin n\pi x/L \, \sin n\pi x_o/L}{(-\rho'\omega^2 + \alpha_T^{\ 2} n^2\pi^2/L^2)} \tag{5.14}$$

where $C_3 = \dfrac{4\pi R_i}{L \, A_c}$

Transverse oscillations of the tube due to the forcing function acting on the entire length of tube is given by

$$y_x = \int_0^L y_{xx_o} \, dx_o \tag{5.15}$$

From Equations (5.14), (5.7) and (5.15) y_x is given by

$$y_x = C_4 \, \Sigma \, b_n \, \sin \frac{n\pi x}{L} \tag{5.16}$$

where $C_4 = \dfrac{4j \, \pi^2 \, R_i \, \rho \, c_o \, I_o \, (KR_i)}{L^2 \, \beta \, A_c}$

$$b_n = \frac{n\left[H_1(1-(-1)^n \cosh(\gamma L)) - G_1(-1)^n \sinh(\gamma L)\right]}{(\gamma^2 + n^2\pi^2/L^2)\,(-\rho'\omega^2 + \alpha^2_{\ T} \, n^2\pi^2/L^2)} \tag{5.17}$$

These details are given in Appendix D.

To predict the radial component of the velocity perturbations in the liquid, the momentum equation in the radial direction, the equation (2.3), is aonsidered. This equation is simplified by neglecting the convective acceleration terms, axial gradients in radial velocity and the radial pressure gradients, because these terms are of the second or smaller order of magnitude as shown in Appendix A. The resulting simplified radial momentum equation is given by the equation.

$$\frac{\partial u'}{\partial t} - \nu\left[\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\ \frac{\partial u'}{\partial r} - \frac{u'}{r^2}\right] = 0 \qquad (5.18)$$

Now, if the radial velocity perturbations in the fluid are assumed to be of the form

$$u'\ (x,\ r,\ t) = u'_r\ (x,r)\ e^{j\omega t}$$

Equation (5.19) can be transformed to the form

$$\frac{\partial^2 u'_r}{\partial r^2} + \frac{1}{r}\frac{\partial u'_r}{\partial r} - \left(\frac{1}{r^2} + \frac{j\omega}{\nu}\right) u'_r = 0 \qquad (5.19)$$

Which is the modified Bessel's equation of order one and, for finite solution at the centre of the tube, the solution of Equation (5.19) can be written as

$$u'_r = g(x,\ j\omega)\ I_1\ (Kr) \qquad (5.20)$$

where $K = \Sigma\sqrt{\frac{j\omega}{\nu}}$

Assuming that the tube remains in contact with the liquid the boundary conditions can be written as

$$u'_r\ (x, r=R_i) = j\omega y_x$$

$$u'_r\ (\ r = 0) = 0 \qquad (5.21)$$

With the above boundary conditions Equation (5.20) reduces to the form

$$u'_r = \frac{I_1(Kr)\ j\omega y_x}{I_1(KR_i)} \qquad (5.22)$$

From Equation (5.2) it can be seen that the magnitude of the radial expansion is not negligible if the pipe line has relatively

low elastic properties. By neglecting the compressibility of the liquid, Equation (2.1) can be written as (Ref. Appendix A)

$$\frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{\partial v'_x}{\partial x} = 0 \qquad (5.23)$$

Substituting Equation (5.16) in Equation (5.22) and then Equation (5.22) in Equation (5.23) the axial flow velocity perturbations at any point x is obtained by integrating the resulting equation w.r.t.x from 0 to x.

$$v'_x - v'_x(0,r) = C_5 \; I_o \; (Kr) \; \Sigma \frac{b_n}{n} \left(\cos \frac{n\pi x}{L} - 1\right) \qquad (5.24)$$

Further it is assumed that $v'_x(0,r) \;=\; 0$

where

$$C_5 = \frac{j\omega K \; C_4 L}{\pi \; I_1(KR_i)}$$

From Equation (3.5) the fluid momentum equation for the axial velocity perturbations is given by

$$\frac{\partial^2 v'_x}{\partial r^2} + \frac{1}{r} \; \frac{\partial v'_x}{\partial r} - \frac{j\omega v'_x}{\nu} - \frac{1}{\mu} \; \frac{dp'_x}{dx} = 0 \qquad (5.25)$$

By going through similar steps as used in Equations (3.6), (3.7) and (3.8) the solution of Equation (5.25) is given by

$$v'_x = b(x, \; j\omega) \; I_o(Kr) - \frac{1}{j\omega\rho} \; \frac{dp'_x}{dx} \qquad (5.26)$$

and the boundary conditions are

at $\quad r = 0 \quad \frac{\partial v'_x}{\partial r} = 0$

$v'_x\ (x, r = 0)$ is given by Equation (5.24)

Substituting above boundary conditions in Equation (5.26), the following Equation is obtained

$$\frac{dp'_x}{dx} = j\omega\rho\left[b(x, j\omega) - C_5\ \Sigma\ \frac{b_n(\cos\frac{n\pi x}{L} - 1)}{n} \right] \qquad (5.27)$$

Then Equation (5.26) reduces to the form

$$v'_x = b(x, j\omega)\ \left\{I_o(Kr) - 1\right\} + C_5\ \Sigma\ \frac{b_n(\cos\frac{n\pi x}{L} - 1)}{n} \qquad (5.28)$$

Average perturbation velocity is obtained by the equation

$$\bar{v}'_x = b(x, j\omega)\ \left[\frac{2}{KR_i}\ I_1\ (KR_i) - 1\right] + C_5\ \Sigma\ \frac{b_n\ (\cos\frac{n\pi x}{L} - 1)}{n} \qquad (5.29)$$

Equation (3.18) can be written as

$$\frac{dp'_x}{dx} = \frac{jB}{\omega}\ \frac{d^2\bar{v}'_x}{dx^2} \qquad (5.30)$$

Substituting Equations (5.27) and (5.29) in Equation (5.30) and rearranging the terms yield

$$\frac{d^2b(x,j\omega)}{dx^2} - \gamma_1^{\,2}\ b(x,j\omega) = C_5\ \gamma_2^{\,2}\ \Sigma\left(\frac{n^2\pi^2}{L^2} - \frac{\omega^2}{c_o^{\,2}}\right)\frac{b_n}{n}\cos\frac{n\pi x}{L} + \frac{\omega^2\gamma_2^{\,2}}{c_o^{\,2}}\ C_5\ \Sigma\ \frac{b_n}{n} \qquad (5.31)$$

where $\Gamma_1^2 = \frac{\Gamma_2^2 \omega^2}{c_o^2}$; $\Gamma_2^2 = \frac{1}{\frac{2}{KR_i} I_1(KR_i) - 1}$

Particular integral of the above nonhomogeneous ordinary differential equation is given by

$$b_p = - C_5 \Gamma_2^2 \sum \frac{(n^2\pi^2/L^2 - \omega^2/c_o^2)}{(\Gamma_1^2 + n^2\pi^2/L^2)} \frac{b_n}{n} \cos \frac{n\pi x}{L}$$

$$-C_5 \frac{\omega^2 \Gamma_2^2}{c_o^2 \Gamma_1^2} \sum \frac{b_n}{n} \tag{5.32}$$

Hence the general solution of the differential equation (5.31) is given by

$$b = G_6 \cosh(\Gamma_1 x) + H_6 \sinh(\Gamma_1 x) + b_p \tag{5.33}$$

If the average velocity perturbation is assumed to be equal to zero at x = 0, then from Equation (5.29)

$$b\ (0,\ j\omega) = 0 \tag{5.34}$$

From Equation (5.33)

$$b(0,\ j\omega) = G_b + b_p\ (0,\ j\omega) = 0$$

or $$G_6 = -b_p\ (0,\ j\omega) \tag{5.35}$$

Substituting Equation (5.32) in Equation (5.35) and rearranging the terms yield

$$G_6 = C_6 \Gamma_2^2 \sum \frac{n\ b_n}{(\Gamma_1^2 + \frac{n^2\pi^2}{L^2})} \tag{5.36}$$

where $$C_6 = \frac{C_5 \pi^2 (1 + \omega^2/(c_o^2 \gamma_1^2))}{L^2}$$

Now by substituting Equations (5.36), (5.32) and (5.33) in Equation (5.29) and rearranging the terms yield

$$\bar{v}'_x = C_6 \cosh \gamma_1 x \, \Sigma \frac{n\, b_n}{(\gamma_1^2 + n^2\pi^2/L^2)} + H_7 \sinh(\gamma_1 x) +$$

$$C_{22} \, \Sigma \frac{b_n \cos n\pi x/L}{n\,(\gamma_1^2 + n^2\pi^2/L^2)} - C_{33} \, \Sigma \frac{b_n}{n} \qquad (5.37)$$

where $H_7 = \dfrac{H_6}{\gamma_2^2}$; $C_{22} = C_5 \, (\gamma_1^2 + \dfrac{2}{c_o^2})$

$$C_{33} = C_5 \, (1 + {}^2/(\gamma_1^2 \, c_o^2))$$

For the pressure perturbations at any point x Equation (3.17) can be written as

$$p'_x = \frac{jB}{\omega} \frac{d\bar{v}'_x}{dx} \qquad (5.38)$$

Substituting Equation (5.37) in Equation (5.38), pressure perturbations is given by

$$p'_x = \frac{jB\gamma_1}{\omega} C_6 \sinh(\gamma_1 x) \, \Sigma \frac{n\, b_n}{(\gamma_1^2 + n^2\pi^2/L^2)} + H_7 \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 x)$$

$$-C_{22} \frac{jB\pi}{\omega L} \, \Sigma \frac{b_n \sin n\pi x/L}{(\gamma_1^2 + n^2\pi^2/L^2)} \qquad (5.39)$$

Now using the boundary conditions

at $x = L$ $\bar{v}'_x = V'_L$ and $p'_x = P'_L$

in Equations (5.37) and (5.39), the following equation is obtained

$$\frac{P'_L}{V'_L} = \left[\frac{jB\gamma_1}{\omega} C_6 \sinh(\gamma_1 L) \, \Sigma \frac{n\, b_n}{(\gamma_1^2 + n^2\pi^2/L^2)} + H_7 \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 L) \right]$$

$$/\left[C_6 \cosh(\gamma_1 L) \, \Sigma \frac{n\, b_n}{(\gamma_1^2 + n^2\pi^2/L^2)} + H_7 \sinh(\gamma_1 L) + C_{22} \, \Sigma \frac{(-1)^n \, b_n}{n(\gamma_1^2 + n^2\pi^2/L^2)} - C_{33} \, \Sigma \frac{b_n}{n} \right] \tag{5.40}$$

Further simplification yields

$$H_7 = \left[\left(\frac{jB\gamma_1}{\omega} C_6 \sinh(\gamma_1 L) - C_6 \frac{P'_L}{V'_L} \cosh(\gamma_1 L) \right) \Sigma \frac{n\, b_n}{(\gamma_1^2 + n^2\pi^2/L^2)} - C_{22} \frac{P'_L}{V'_L} \, \Sigma \frac{(-1)^n \, b_n}{n(\gamma_1^2 + n^2\pi^2/L^2)} + C_{33} \frac{P'_L}{V'_L} \, \Sigma \frac{b_n}{n} \right] \Big/ \left[\frac{P'_L}{V'_L} \sinh(\gamma_1 L) - \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 L) \right] \tag{5.41}$$

The frequency response of the velocity together with the effect of the transverse waves along the pipe line can be obtained by adding Equations (3.22) and (5.37).

$$\overline{V} = \overline{V}_x + \overline{v}'_x \tag{5.42}$$

By substituting the constant H_7 in Equation (5.37) and then substituting Equations (5.37) and (3.22) in Equation (5.42) yield

$$\bar{V} = G_1 \Big[- C_6 \cosh(\Gamma_1 x) S_5 + \sinh(\Gamma_1 x)(-C_9 S_5 + C_{22} \frac{P'_L}{V'_L} S_6 -$$

$$C_{33} \frac{P'_L}{V'_L} S_7) / C_{10} - C_{22} \Sigma \frac{(-1)^n \cos n\pi x/L \sinh \Gamma L}{c_n}$$

$$+ C_{33} S_7 + K_2 \cosh(\Gamma x) \Big]$$

$$+ H_1 \Big[C_6 \cosh(\Gamma_1 x) S_8 + \sinh(\Gamma_1 x) \; (C_9 S_8 -$$

$$C_{22} \frac{P'_L}{V'_L} S_9 + C_{33} \frac{P'_L}{V'_L} S_{10}) / C_{10}$$

$$+ C_{22} \Sigma \frac{\left(1-(-1)^n \cosh \Gamma L\right)}{c_n} \cos \frac{n\pi x}{L} - C_{33} S_{10} +$$

$$K_2 \sinh(\Gamma x) \Big] \tag{5.43}$$

or

$$\bar{V} = C_{nx5} \; G_1 + C_{nx6} \; H_1 \tag{5.44}$$

where $C_{nx5} = -C_6 \cosh(\Gamma_1 x) S_5 + C_{n1} \sinh(\Gamma_1 x) - C_{22} \cdot$

$$\Sigma \frac{(-1)^n \sinh(\Gamma L) \cos(n\pi x/L)}{c_n} + C_{33} S_7 + K_2 \cosh(\Gamma x)$$

$$C_{nx6} = C_6 \cosh(\Gamma_1 x) S_8 + C_{n2} \sinh(\Gamma_1 x) + C_{22} \cdot$$

$$\Sigma \frac{\left(1-(-1)^n \cosh(\Gamma L)\right)}{c_n} \cos \frac{n\pi x}{L} - C_{33} S_{10} + K_2 \sinh(\Gamma x)$$

$$C_{n1} = \frac{-C_9 S_5 + C_{22} \frac{P'_L}{V'_L} S_6 - C_{33} \frac{P'_L}{V'_L} S_7}{C_{10}}$$

$$C_{n2} = \frac{C_9 \; S_8 - C_{22} \frac{P'_L}{V'_L} S_9 + C_{33} \frac{P'_L}{V'_L} S_{10}}{C_{10}}$$

$$C_9 = \frac{jB\gamma_1}{\omega} \; C_6 \; \sinh(\gamma_1 L) - C_6 \frac{P'_L}{V'_L} \cosh(\gamma_1 L)$$

$$C_{10} = \frac{P'_L}{V'_L} \sinh(\gamma_1 L) - \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 L)$$

$$S_5 = \Sigma \frac{n^2 \; (-1)^n \; \sinh(\gamma L)}{c_n} \; ; \; S_6 = \Sigma \frac{\sinh(\gamma L)}{c_n} \; ; S_7 = \Sigma \frac{(-1)^n \sinh(\gamma L)}{d_n}$$

$$S_8 = \Sigma \frac{n^2(1-(-1)^n \; \cosh(\gamma L))}{c_n} \; ; \; S_9 \; \Sigma \frac{(-1)^n \; \cosh(\gamma L)}{c_n} \; ;$$

$$S_{10} = \Sigma \frac{1-(-1)^n \; \cosh(\gamma L)}{d_n} \; ; \quad d_n = (-\rho'\omega^2 + \alpha_T^2 \; n^2\pi^2/L^2)(\gamma^2 + n^2\pi^2$$

$$C_n = (-\rho'\omega^2 + \alpha_T^2 \; n^2\pi^2/L^2) \; (\gamma^2 + n^2\pi^2/L^2)(\gamma_1^2 + n^2\pi^2/L^2)$$

at $\quad x = 0$

$$C_{n05} = K_2 \; ; \qquad C_{n06} = 0$$

at $x = L$

$$C_{nL5} = -C_6 \; S_5 \; \cosh(\gamma_1 L) + C_{n1} \; \sinh(\gamma_1 L) - C_{22} \; S_6 + C_{33} \; S_7 + K_2 \; \cosh(\gamma L)$$

$$C_{nL6} = C_6 \; S_8 \; \cosh(\gamma_1 L) + C_{n2} \; \sinh(\gamma_1 L) + C_{22} \; S_9 - C_{33} S_{10} + K_2 \; \sinh(\gamma L)$$

Frequency response of the output pressure including the effect of the transverse waves along the tube can be

obtained by adding Equations (3.23) and (5.39)

$$\overline{P} = P_x + p'_x \tag{5.45}$$

By substituting the constant H_7 in Equation (5.39) and then substituting Equations (3.23) and (5.39) in Equation (5.45) yield

$$\begin{aligned}\overline{P} = G_1 \Big[& -\frac{jB\Gamma_1 C_6}{\omega} \sinh(\Gamma_1 x) S_5 + \frac{jB\Gamma_1 \cosh(\Gamma_1 x)}{\omega} \times \\ & (-C_9 S_5 + C_{22} \frac{P'_L}{V'_L} S_6 - C_{33} \frac{P'_L}{V'_L} S_7) / C_{10} \\ & + C_{22} \frac{jB\pi}{\omega L} \Sigma \frac{n(-1)^n \sinh(\Gamma L) \sin n\pi x/L}{c_n} + C_8 \sinh(\Gamma x)\Big] \\ & + H_1 \Big[\frac{jB\Gamma_1 C_6}{\omega} \sinh(\Gamma_1 x) S_8 + \frac{jB\Gamma_1 \cosh(\Gamma_1 x)}{\omega} \times \\ & (C_9 S_8 - C_{22} \frac{P'_L}{V'_L} S_9 + C_{33} \frac{P'_L}{V'_L} S_{10}) / C_{10} \\ & - C_{22} \frac{jB\pi}{\omega L} \Sigma \frac{n(1-(-1)^n \cosh(\Gamma L))}{c_n} \sin \frac{n\pi x}{L} + C_8 \cosh(\Gamma x)\Big]\end{aligned} \tag{5.46}$$

or $$\overline{P} = C_{nx7} G_1 + C_{nx8} H_1 \tag{5.47}$$

where

$$\begin{aligned}C_{nx7} = & -\frac{jB\Gamma_1}{\omega} C_6 \sinh(\Gamma_1 x) S_5 + C_{n1} \frac{jB\Gamma_1}{\omega} \cosh(\Gamma_1 x) \\ & + C_{22} \frac{jB\pi}{\omega L} \Sigma \frac{n(-1)^n \sinh(\Gamma L) \sin n\pi x/L}{c_n} + C_8 \sinh(\Gamma x)\end{aligned}$$

$$C_{nx8} = + \frac{jB\gamma_1}{\omega} C_6 \sinh(\gamma_1 x) S_8 + \frac{jB\gamma_1}{\omega} C_{n2} \cosh(\gamma_1 x)$$

$$-C_{22} \frac{jB\pi}{\omega L} \Sigma \frac{n(1-(-1)^n \cosh(\gamma L)\sin n\pi x/L}{c_n} + C_8 \cosh(\gamma x$$

at $\quad x = 0$

$$C_{n07} = \frac{jB\gamma_1}{\omega} C_{n1} \; ; \quad C_{n08} = \frac{jB\gamma_1}{\omega} C_{n2} + C_8$$

at $\quad x = L$

$$C_{nL7} = -\frac{jB\gamma_1}{\omega} C_6 \; S_5 \sinh(\gamma_1 L) + C_{n1} \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 L) + C_8 \sinh(\gamma L)$$

$$C_{nL8} = \frac{jB\gamma_1}{\omega} C_6 \; S_8 \sinh(\gamma_1 L) + C_{n2} \frac{jB\gamma_1}{\omega} \cosh(\gamma_1 L) + C_8 \cosh(\gamma L)$$

The velocity and pressure response including the effect of transverse waves along the tube are given by Equations (4.47), (4.48) and (4.49) in terms of above constants as

$$\frac{P_L + P''_L}{P_o + P'_o} = \frac{V_o + V'_o}{P_o + P'_o} \frac{C_{nL7}}{K_2} + \left[\frac{1}{C_{n08}} - \frac{V_o + V'_o}{P_o + P'_o} \frac{C_{n07}}{C_{n08} K_2} \right] C_{nL8} \tag{5.48}$$

$$\frac{V_L + V'_L}{V_o + V'_o} = \frac{C_{nL5}}{K_2} + \left[\frac{P_o + P'_o}{(V_o + V'_o) C_{n08}} - \frac{C_{n07}}{C_{n08} \; K_2} \right] C_{nL6} \tag{5.49}$$

and

$$\frac{P_o + P'_o}{V_o + V'_o} = \left[\frac{V_L + V'_L}{P_L + P'_L} \left(\frac{C_{nL7}}{K_2} - \frac{C_{nL8}}{K_2} \frac{C_{n07}}{C_{n08}} \right) + \frac{C_{n07}}{K_2} \frac{C_{nL6}}{C_{n08}} - \frac{C_{nL5}}{K_2} \right]$$

$$\Big/ \left[\frac{C_{nL6}}{C_{n08}} - \frac{V_L + V'_L}{P_L + P'_L} \frac{C_{nL8}}{C_{n08}} \right] \tag{5.50}$$

Where $(P_L + P'_L)/(V_L+V'_L)$ is the load impedence and a method of estimating the load impedence is given in Appendix E. The frequency response of the system including the effect of transverse waves can be estimated from Equations (5.48) and (5.49) with the appropriate system parameters.

CHAPTER VI

COMPUTED RESULTS AND DISCUSSION

6.1 Computer Results:

The physical characteristics given in Table 6.1[14] are used for the computation of theoretical results.

Table 6.1

Tygon tube	
Length L	10.8 ft
Inside diameter, $2R_i$	0.1875 inch
Outside diameter, $2R_o$	0.3125 inch
Effective bulk modulus, B	110.1 lb/in^2
Density ρ'	1.3 x 62.4 lb/st^3
Poission's Ratio σ	0.45
Load	
Area of U-tube, A_L	0.03267 in^2
Mass of mercury column, m_L	0.3166 lb
Spring constant due to gravity, per unit area, K_S	0.98 lb/in^3
Water	
Density,	62.4 lb/ft^3
Isentropic phase velocity c_o	90.4 ft/sec
Kinematic viscosity	0.00155 in^2/sec

Computer programs for solving the pertinent equations, including the programs for the modified Bessel's function and its asymptotic form, were written for the IBM 7044 digital computer.

To calculate the velocity and pressure response with and without the tube effects, Equations (3.29), (3.30), (4.51), (4.52), (5.48) and (5.49) are used. Velocity and pressure response without the tube effect were calculated to know the magnitude of the effects of the tube on the dynamic flow of the liquid. Calculations were made for different values of the damping constant of the tube material ($K + \frac{4}{3}\eta$) and the viscous damping of the fluid. Knowing the effective bulk modulus, modulus of elasticity of the tube material is calculated from Equation (2.10). A method of estimating the load impedence when the load consists of only an inertia with the restoring force obtained from the gravitational field is shown in Appendix E and the resulting load impedence is given by the equation

$$\frac{P_L}{V_L} = j\omega\left[m_L \frac{A_p}{A_L{}^2} - \frac{K_S A_p}{A_L}\right] \qquad (6.1)$$

it is further assumed that $\frac{P_L}{V_L} = \frac{P_L'}{V_L'}$

As the experimental results can only be correlated to the theoretical results by suitably adjusting some parameters like the visco-elastic damping of the tube material, Coulomb friction, viscous damping etc. the theoretical results are to be calculated for several values of these parameters. For some selected parameters the frequency response of the liquid with and without the tube effects are given in Figures 6.1 to 6.14. Some Figures only the effect of the tube on the dynamic response of the liquid.

6.2 General Discussion:

For the theoretical analysis of the dynamic response of liquid flow in pipe lines, linear forms of the basic equation of motion have been considered. If the liquid flow in smooth and uniform circular tube is considered, it is reasonable to assume that the fluid flow exhibits axial symmetry. As discussed in Appendix A, the fluid flow equations are said to be linear when the disturbances are small in which case the convective acceleration terms and the terms consisting of the bulk viscosity reduce to extremely small order of magnitude. It has also been shown in this appendix that for small diameter tubes the radial flow velocity and the radial pressure gradients are of very small order of magnitude. The radial momentum equation has been used to include the effect of the transverse waves along a soft elastic tube on the dynamic response of the liquid. The energy equation has not been taken into consideration in the entire analysis because of the negligible changes in the temperature and, hence, in the internal energy of liquids due to their high specific heats, low compressibility and negligible heat transfer to the pipe line system when under the action of viscous dissipation.

For small disturbances in flow conditions, it is reasonable to assume that the equation of state represented by Equation (2.4) is isentropic since the dissipation effects in the liquid are very small, the specific heats of the liquid and the tube material are very high and the energy changes

are quite small. Equation (2.4) is a linear form of the equation for the state and the isentropic phase velocity is given by the linear Equation (2.9).

It can be seen from Appendix A the assumptions that the variations of inside radius and the radial velocities are very small are not entirely justified in the case of fluid flows in soft elastic tubes. Though these quantities are resonably small they could still introduce some error in the analysis. But by indluding the effects in the analysis, the system would become quite complicated thus making an analytical study almost impossible. Also, in this appendix the form of solutions (A.21a) and (A.21b) make it possible to reduce the equations of motion for the tube material to a form suitable for the theoretical analysis. Once this is accomplished the terms involving the radial gradients vanish and for further analysis it is assumed that the form of solution given by Equations (4.3) and (5.3) are best suited since the variations in the tube radius are quite small.

6.3 Discussion of Results:

Figures (6.1) to (6.14) show the theoretical computed results from Equations (3.29), (3.30), (4.51), (4.52), (5.48) and (5.49). These figures show that the effect of the longitudinal waves on the dynamic response of the liquid are very small compared to the effect of the transverse waves along the tube. This is understandable as the shear drag oscillations on the tube walls in the case of longitudinal wave are small because of the low viscosity of the liquid.

From the above figures it can be observed that the effect of tube reduces as the visco-elastic damping constant increases for a particular value of the viscosity of the liquid. It is due to the fact that, as the damping constant increases a greater portion of the forces on the tube walls is dissipated and hence the reduction of the dynamic effects of the tube on the liquid flow. It can also be observed that the tube is showing a maximum effect around the resonant frequencies of the liquid flow, which may be due to the maximum effect of the forcing function acting on the tube at these frequencies. The effect of the tube increases, for a particular value of visco-elastic damping, with the increase of viscosity of the liquid, which is due to the increase in the effect of the forcing function acting on the tube walls with the increase of the viscosity of the liquid. Around the resonant frequencies of the tube, the effect of the waves along the tube does not show considerable effects on the dynamic response of the

liquid. This may be, because of the low value of the forcing function at the resonant frequencies of the tube owing to the dynamics of the main flow. For low values of the visco-elastic damping constant, the effect of the transverse waves are not shown. At these values the effect of the tube turn out to be very large, hence for these values this analysis is not valid. The effect of the tube reduces with the increase of the frequency.

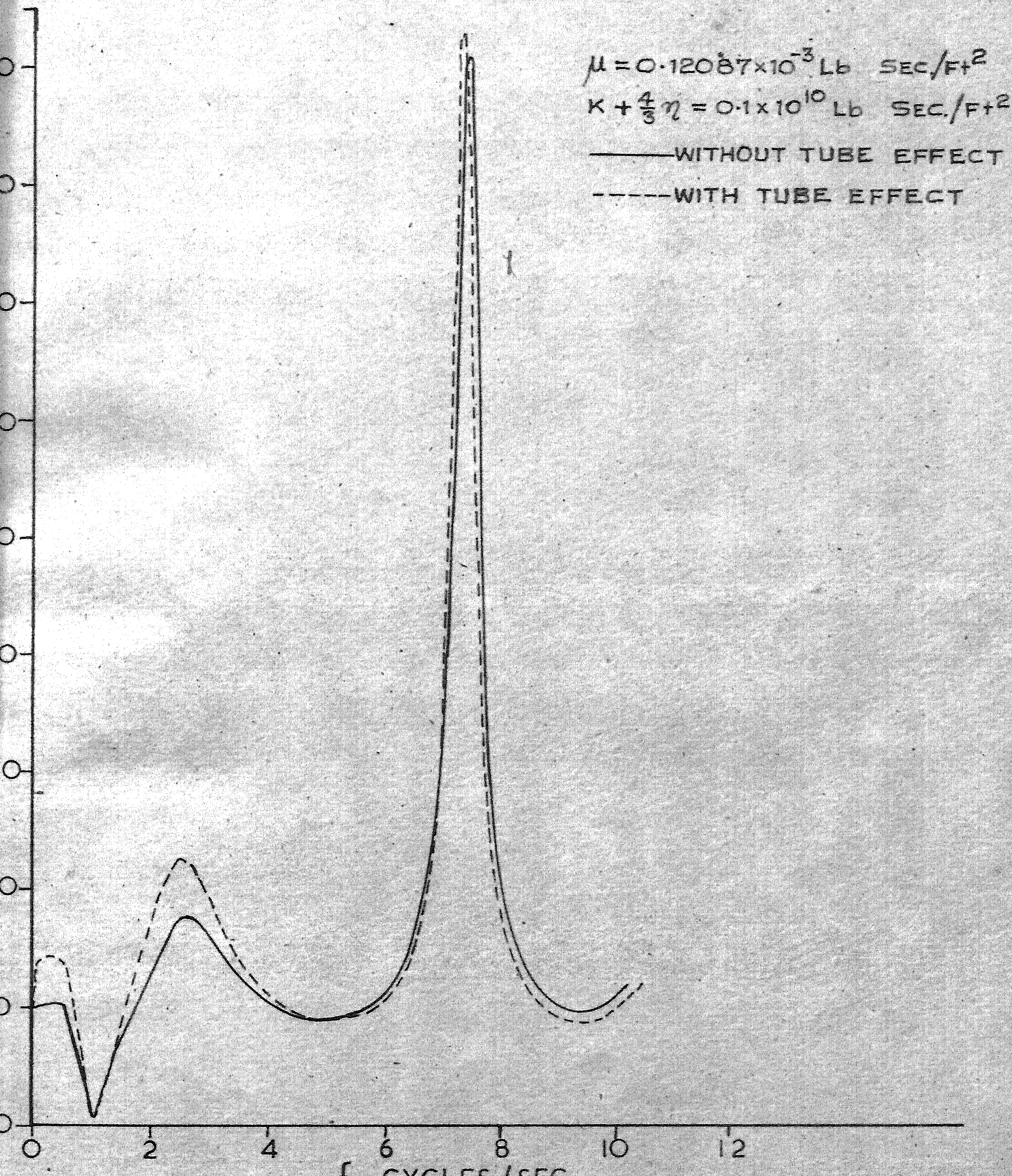

FIG. 6·1 EFFECT OF TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID· (PRESSURE)

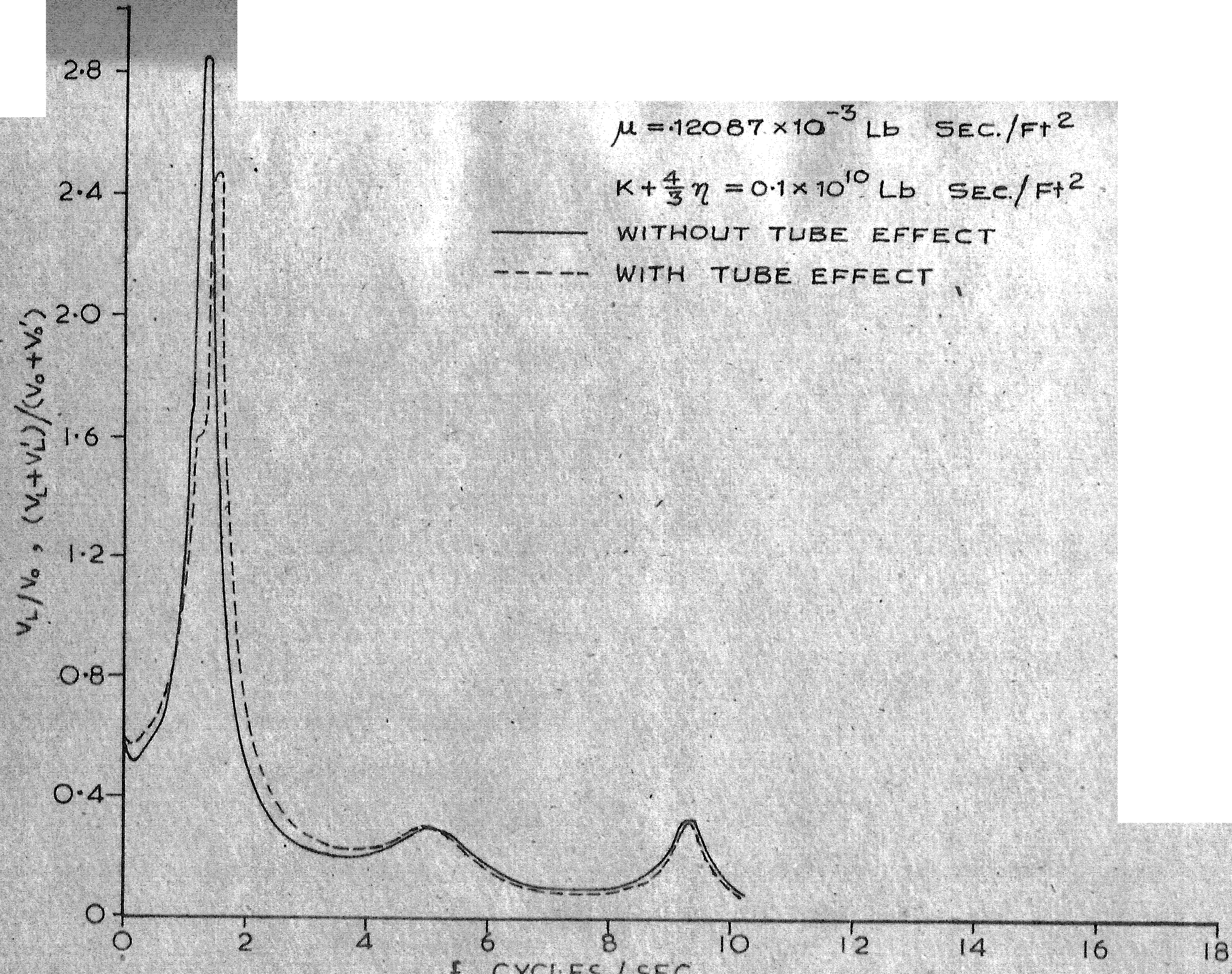

FIG. 6·2 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (VELOCITY)

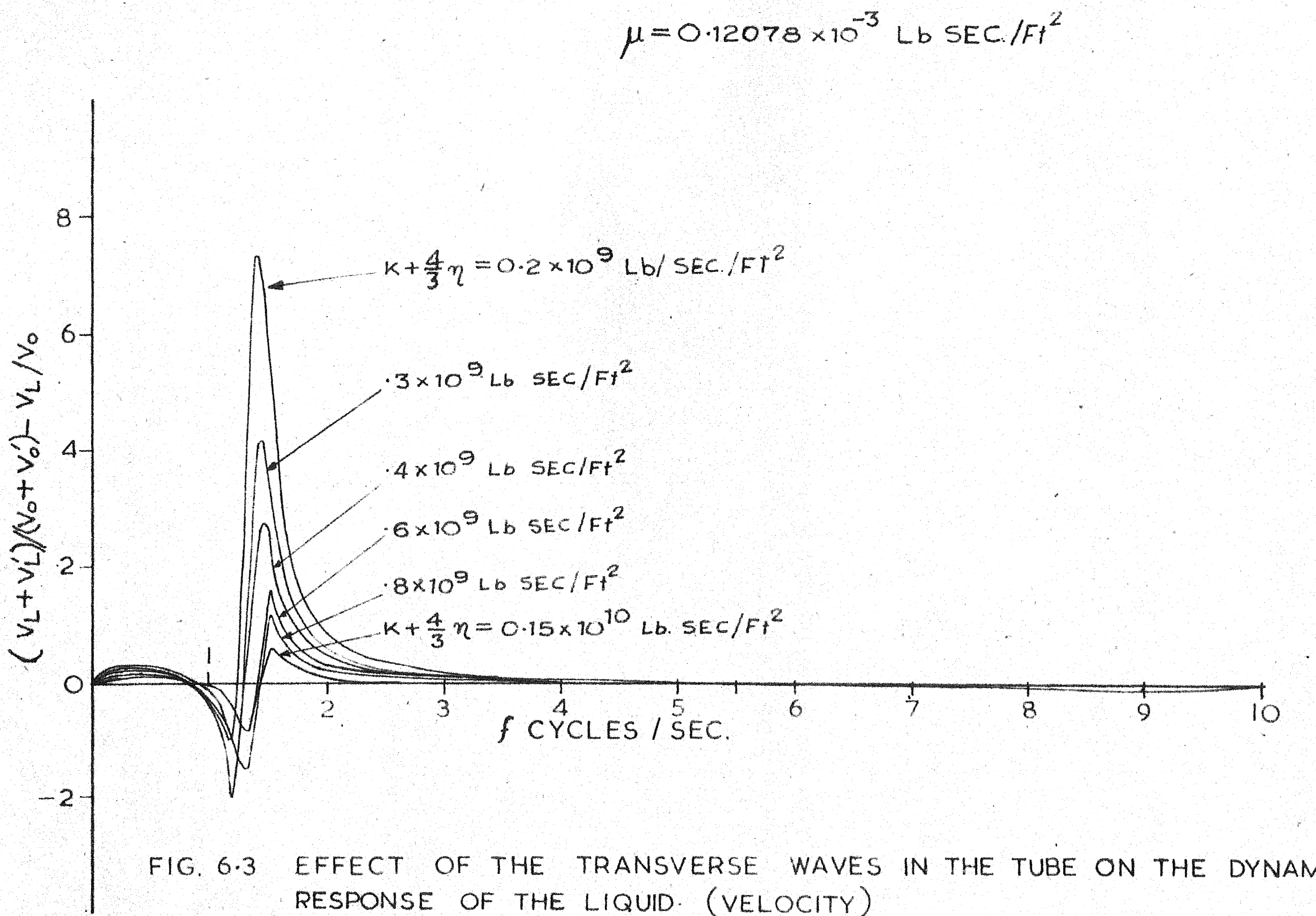

FIG. 6·3 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (VELOCITY)

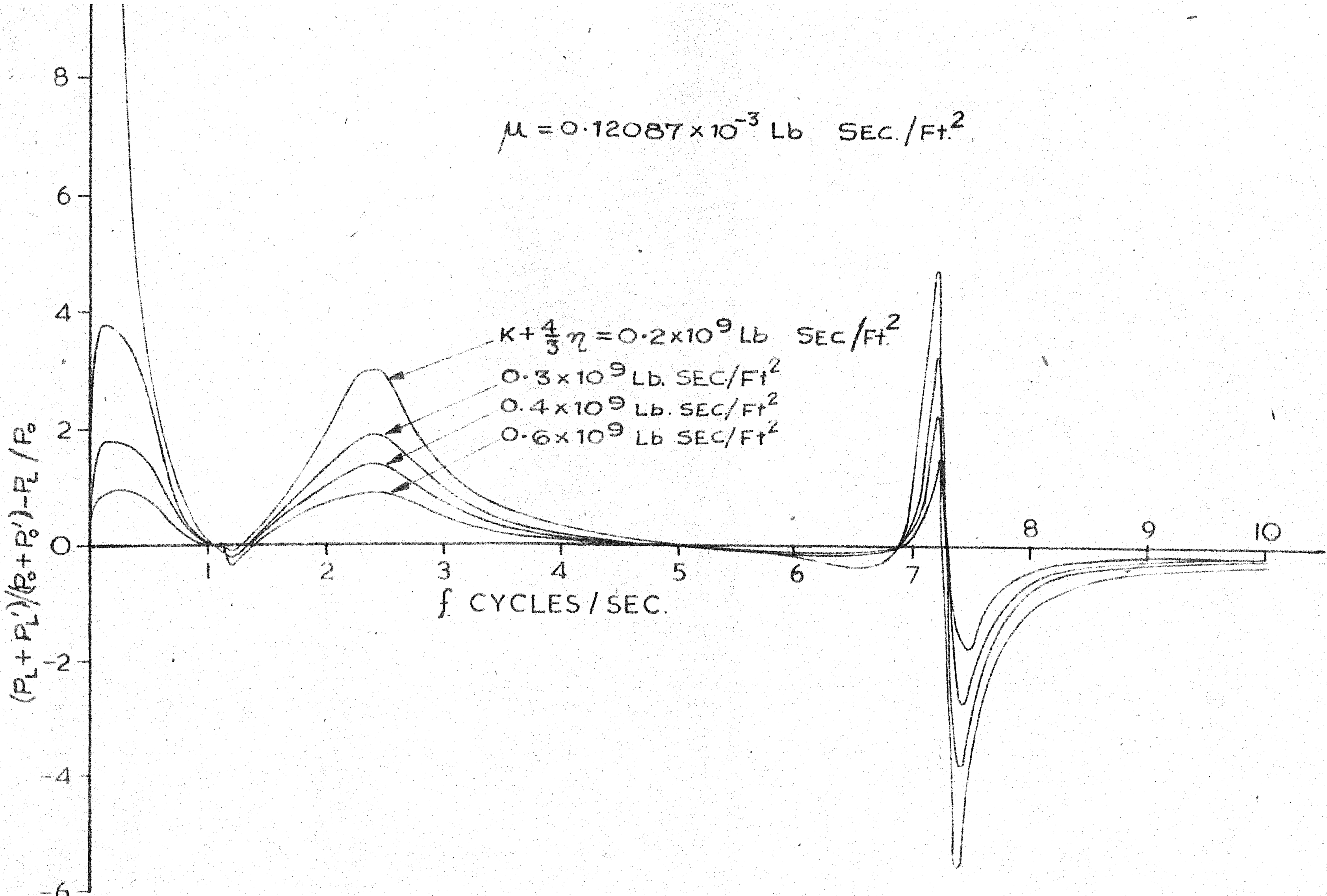

FIG. 6·4 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (PRESSURE)

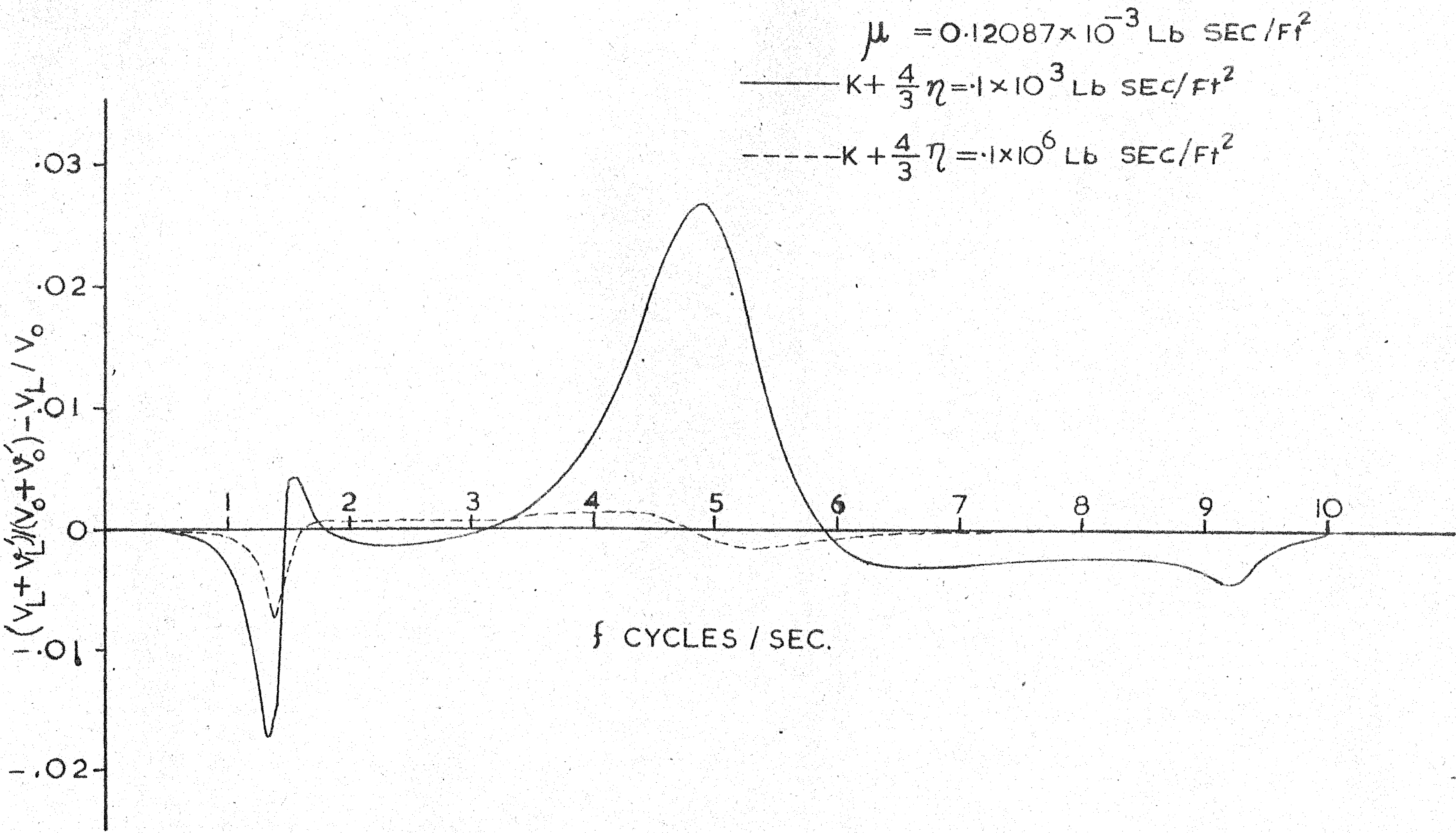

FIG. 6.5 EFFECT OF LONGITUDINAL WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (VELOCITY)

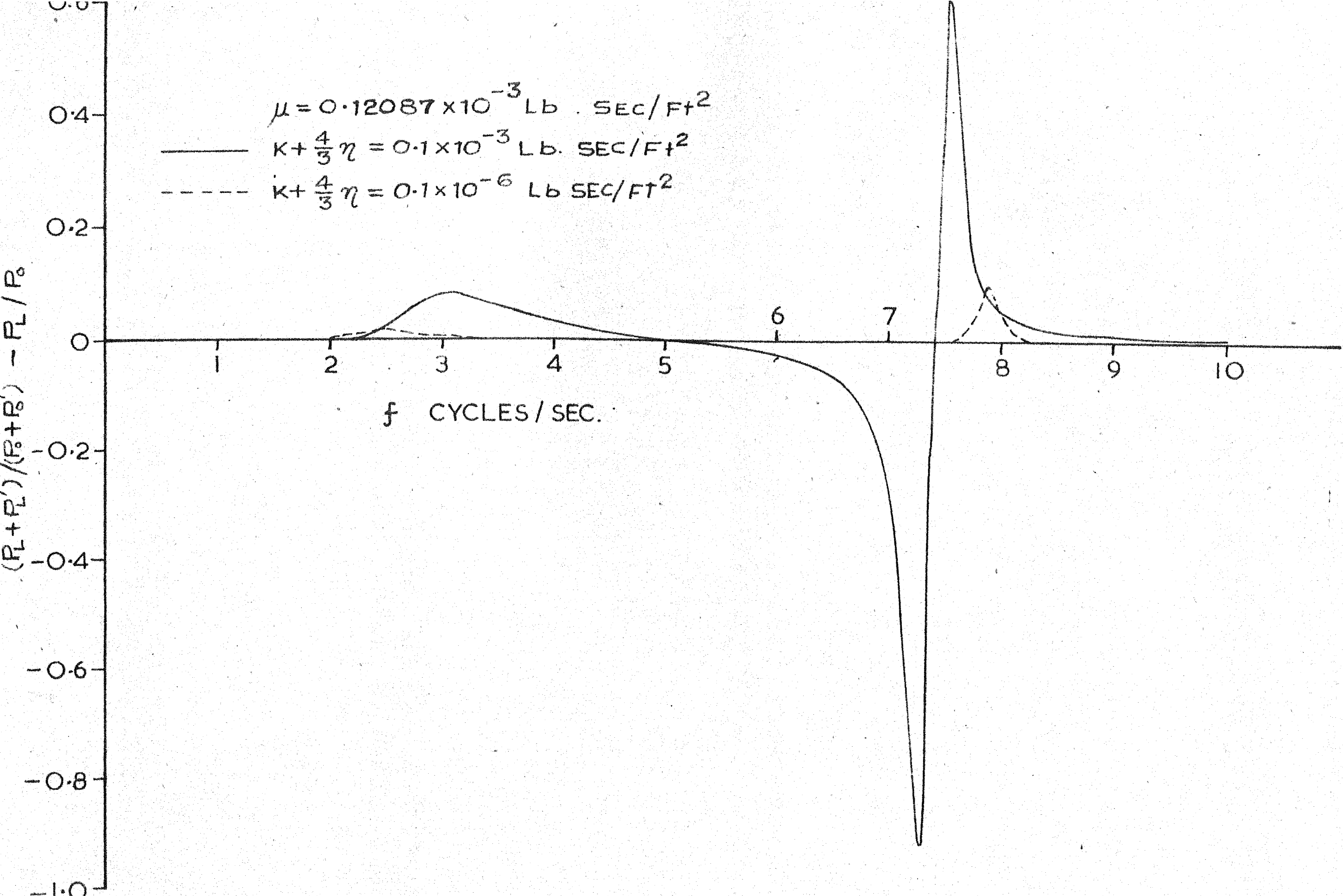

FIG. 6·6 EFFECT OF THE LONGITUDINAL WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (PRESSURE)

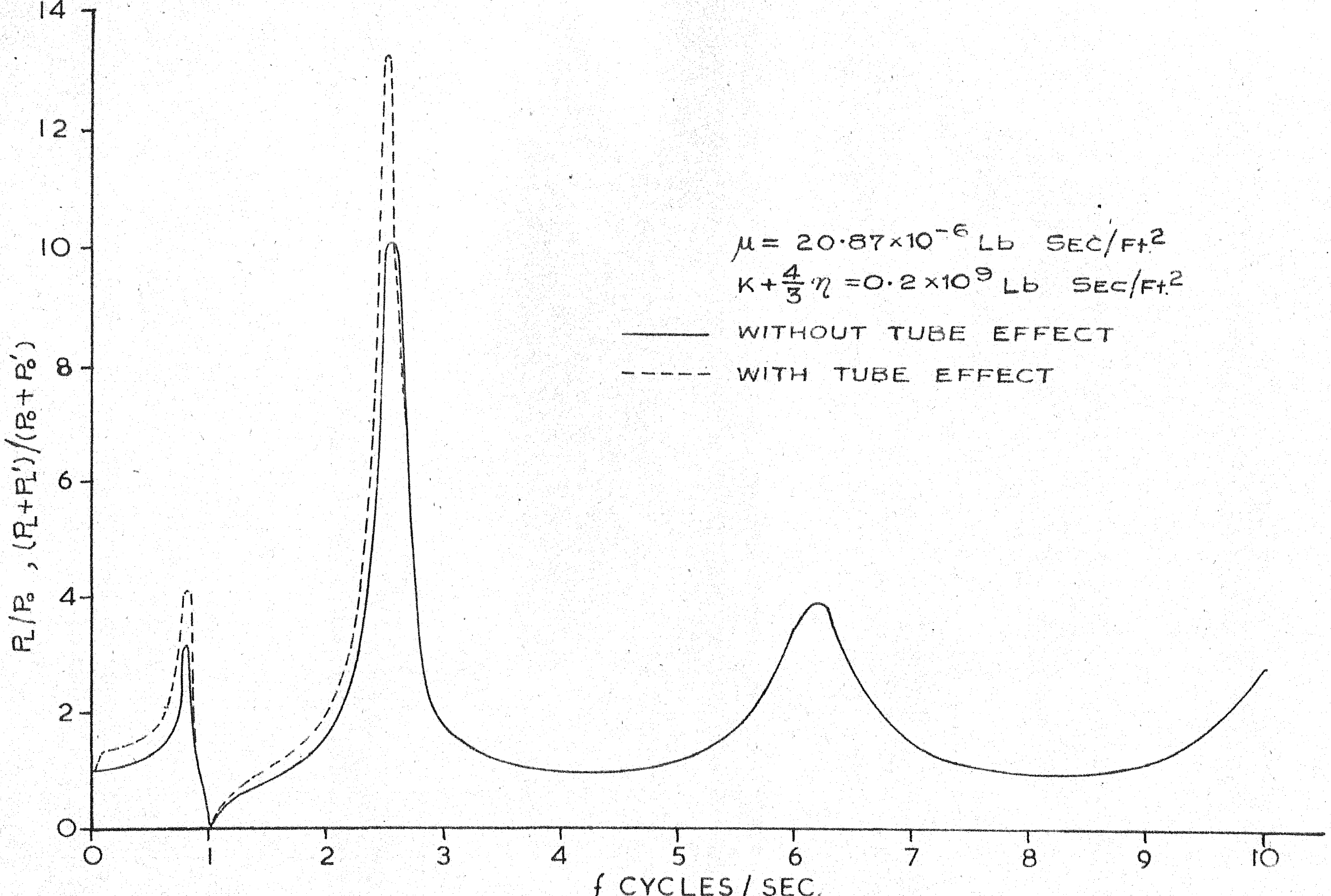

FIG. 6·7 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (PRESSURE)

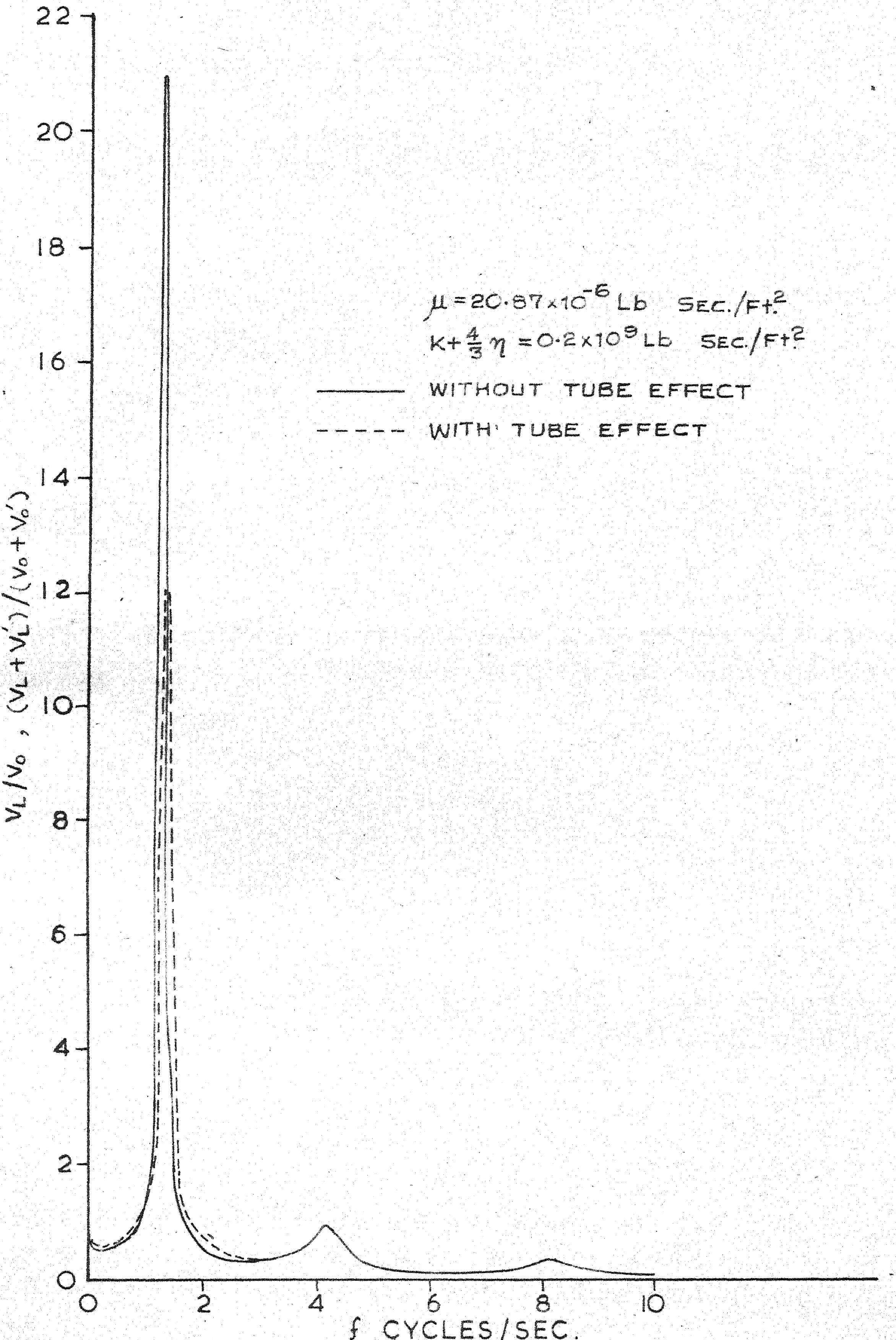

FIG. 6·8 EFFECT OF TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (VELOCITY)

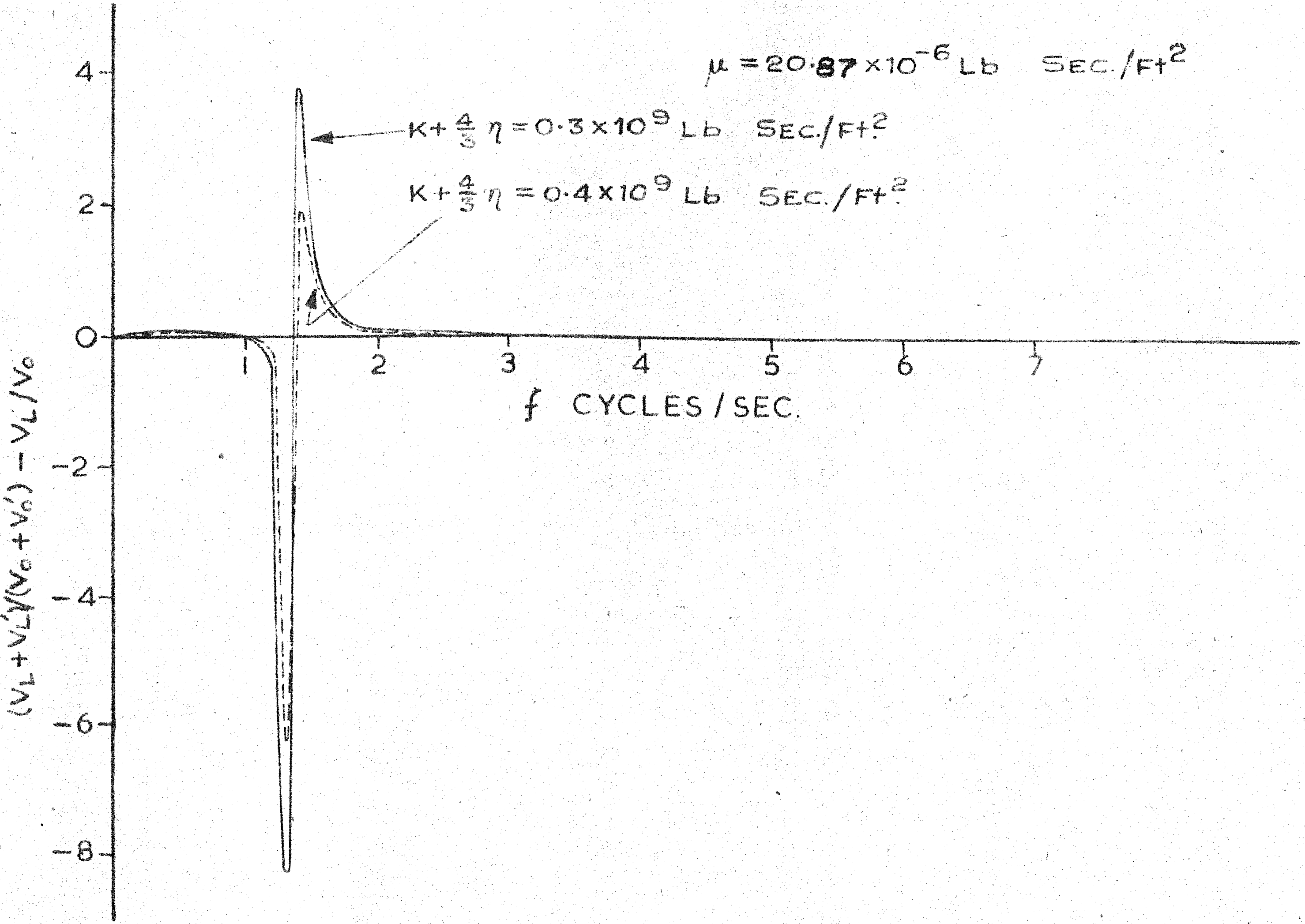

FIG. 6·9 EFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (VELOCITY)

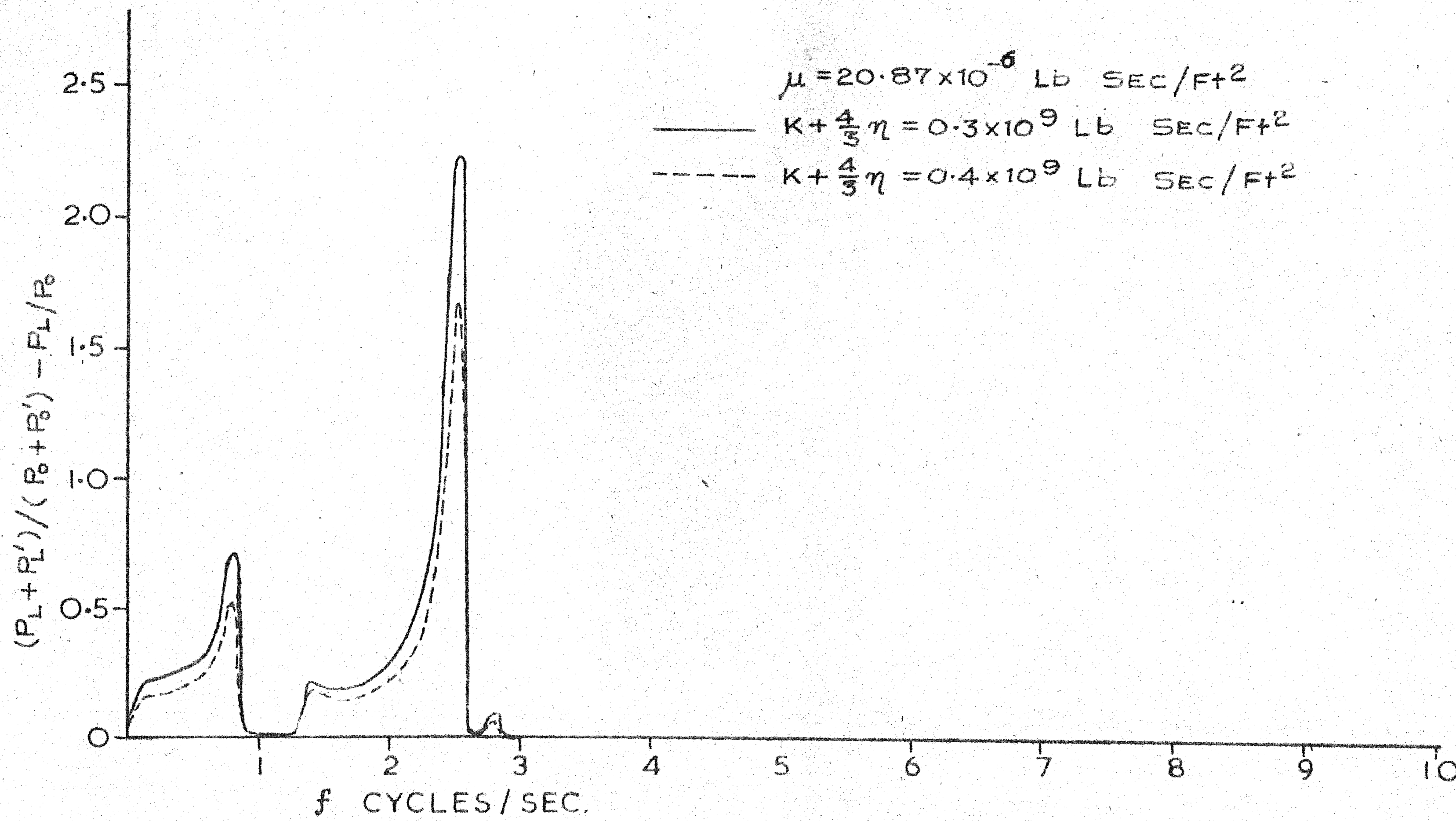

FIG. 6.10 EFFECT OF TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (PRESSURE)

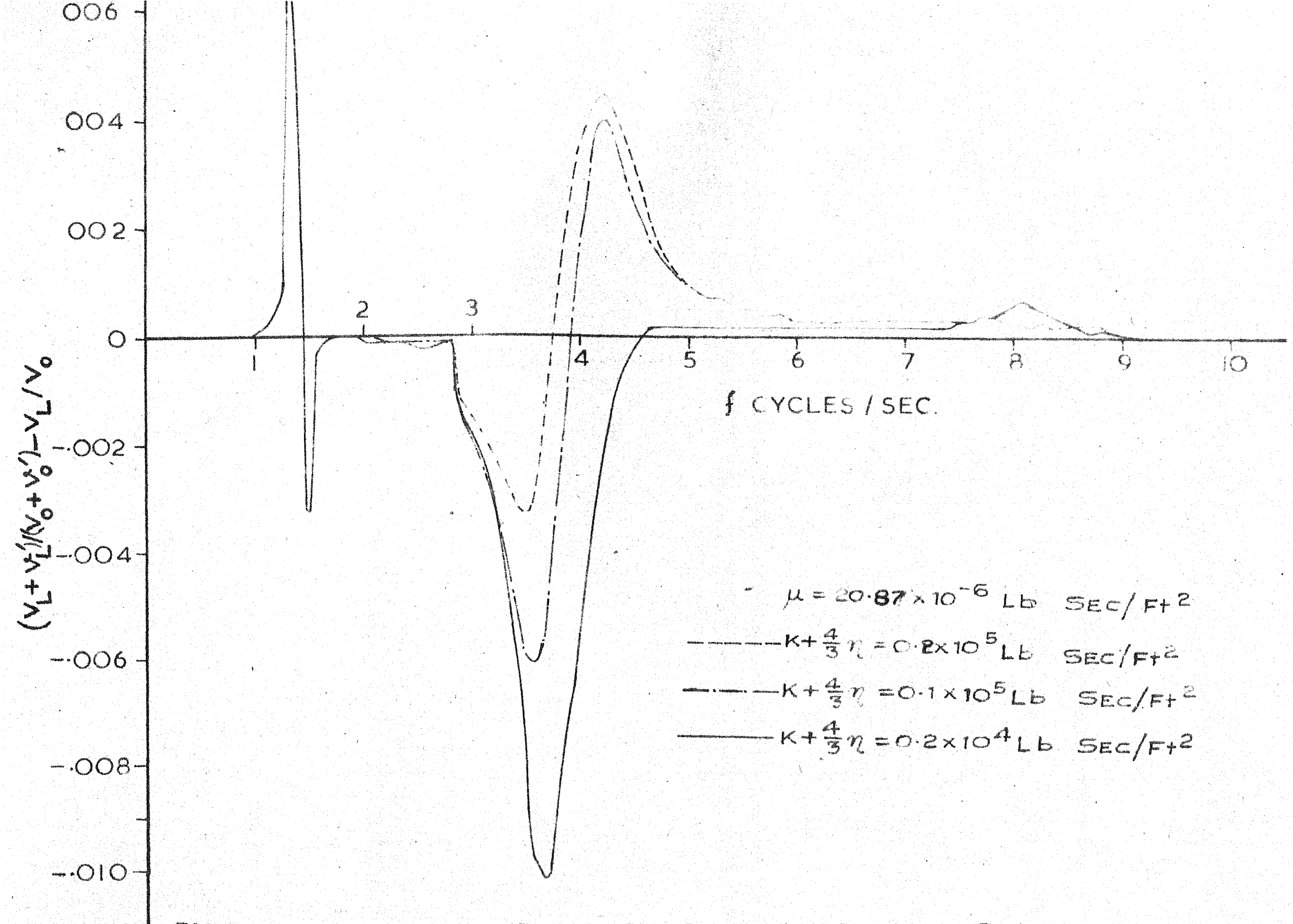

FIG. 6·11 EFFECT OF THE LONGITUDINAL WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID (VELOCITY)

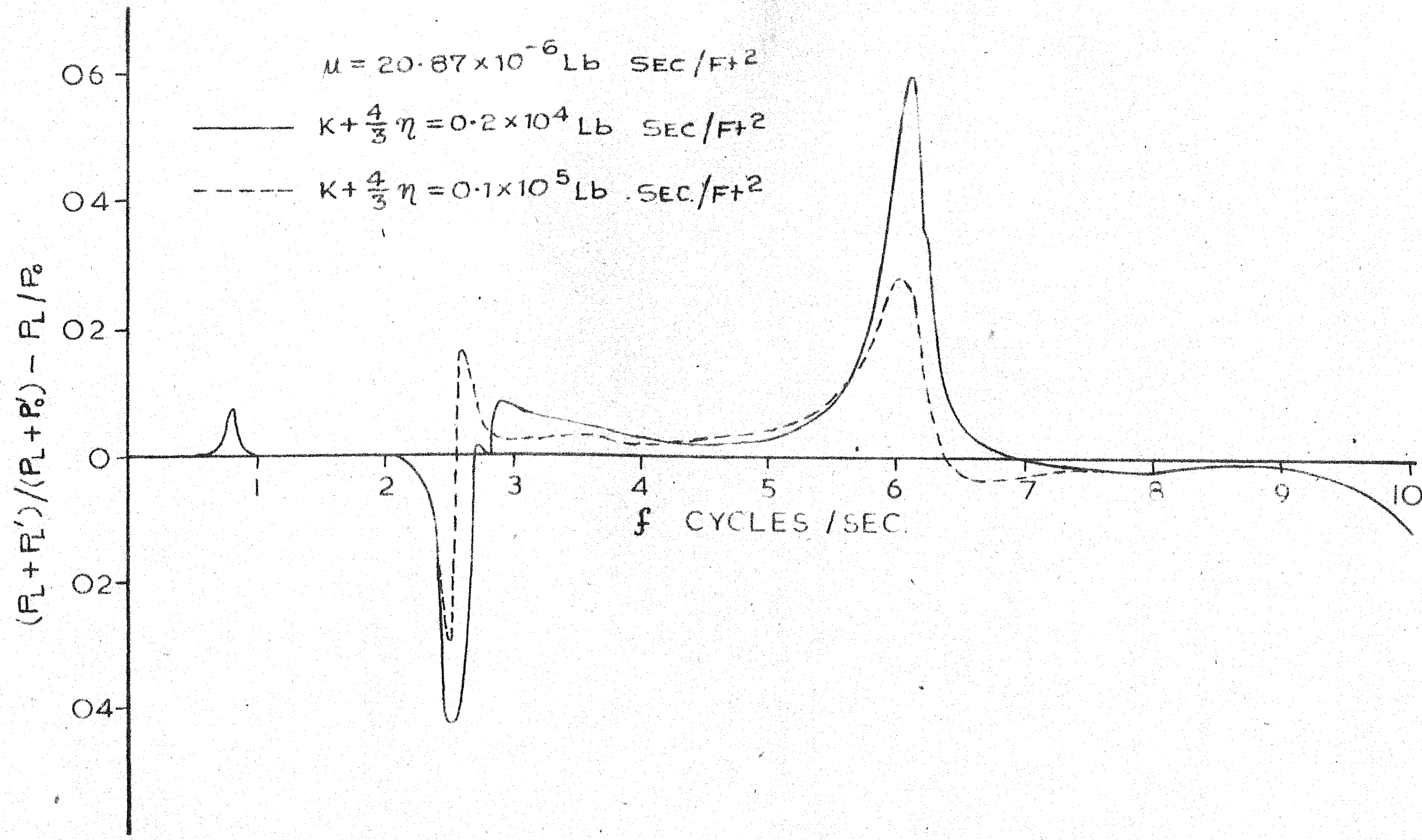

FIG. 6·12 EFFECT OF THE LONGITUDINAL WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (PRESSURE)

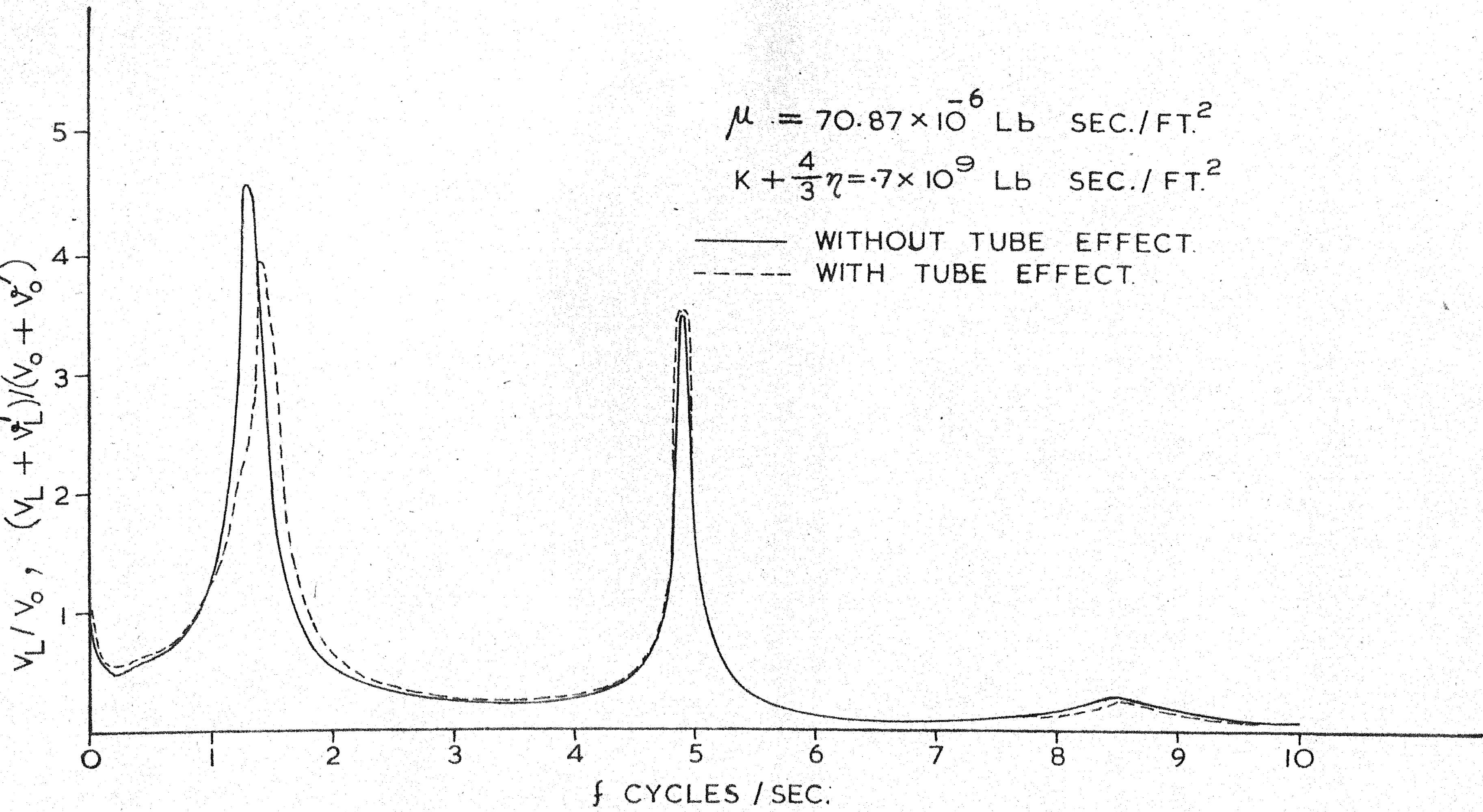

FIG.6.13 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (VELOCITY)

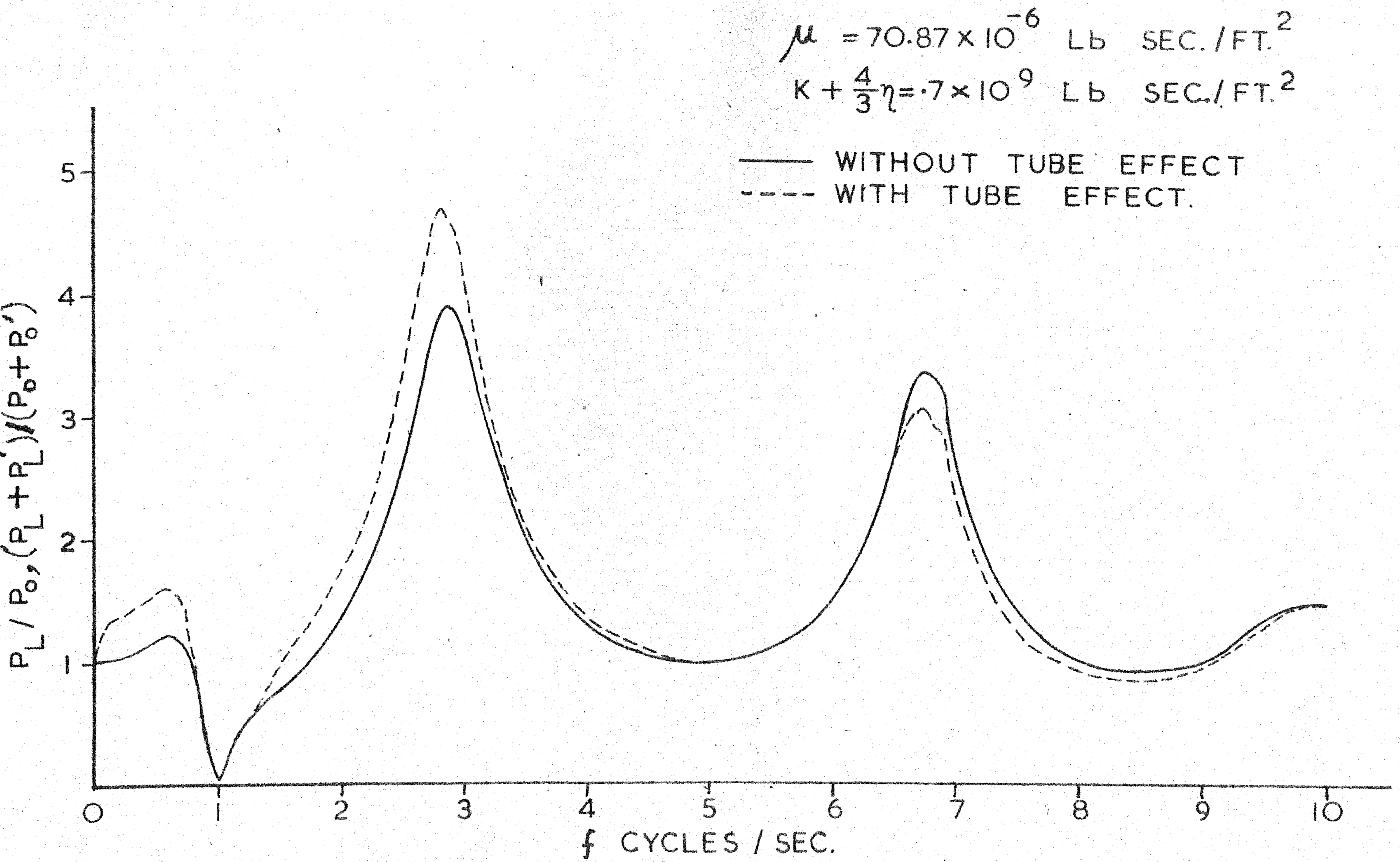

FIG. 6.14 EFFECT OF THE TRANSVERSE WAVES IN THE TUBE ON THE DYNAMIC RESPONSE OF THE LIQUID. (PRESSURE)

REFERENCES

1. Kirchoff, G., Presented in Theory of Sound by J.W.Strutt, Vol.II, 1896, Mac Millan and Co. Ltd., p. 319.

2. Strutt, J.W., (Baron Rayleigh), The Theory of Sound, Vol. II, 1896, Mac Millan and Co. Ltd., p.315.

3. Joukowsky, N., 'Water Hammer' proceedings of the American Water Works Association, 1904, p. 341.

4. Allievi, L., Presented in Engineering Fluid Mechanics by C.Jaegar, Blackie and Sons Limited,London, 1956, p. 275.

5. Wood, F.M., 'The Application of Heavyside's operational Calculus to the Solution of Problems in Water Hammer', Trans. ASME, Vo. 59, 1937, p. 707.

6. Rich, G.R., 'Water Hammer Analysis by Laplace-Mellin Transformation', Trans. ASME, Vol. 67, 1945,p.361.

7. Iberall, A.S., 'Attenuation of Oscillatory Pressure in Instrument Lines', Journal of Research, National Bureau of Standards, Vol. 45, July 1950, RP 2115, p. 85.

8. Nichols, N.B., 'The Linear Properties of Pneumatic Transmission Lines', ISA Transactions, Vol. 1, No. 1, January 1962.

9. Thomson, W.T., 'Transmission of Pressure Waves in Liquid Filled Tubes', Proceedings of the First U.S. National Congress of Applied Mechanics, 1951, p.927.

10. Morgan, G.W. and Kiely, J.P., 'Wave Propagation in a Viscous Liquid Contained in a Flexible Tube,' Journal of the Accoustic Society of America, Vol. 26, 1954, p. 323.

11. Gerlach, G.R. and J.D.Parker, 'Wave Propagation in Viscous fluid lines Including High Mode Effects' Journal of Basic Engineering, Trans. ASME, Series D, Vol. 89, Dec. 1967, p. 782.

12. Brown, F.T., 'The Transient Response of Fluid Lines', Journal of Basic Engineering, Trans. ASME, Series D, Vol. 84, 1962, p. 547.

13. D'Souza, A.F. and Oldenburger, R., 'Dynamic Response of Fluid Lines', Journal of Basic Engineering, Trans. ASME, Series D, Vol. 86, 1964, p. 589.

14. Srinivas, V., 'A Study of the Dynamic Response of the Flow of Liquids in pipe line Systems', Ph.D. Thesis, University of Saskachwan, 1965.

15. Hermann Schlichting, 'Boundary Layer Theory', Translated by J.Kestin, McGraw-Hill Book Co., Inc.,1960,p.53.

16. Den Hartog, J.P., 'Advanced Strength of Materials', McGraw-Hill, 1952, p. 57.

17. Martin Redwood, 'Mechanical Wave Guides' Pergmon Press, 1960, p.13.

18. Fredrickson, 'Principles and Applications of Rheology', Prentice-Hall Inc., 1964, p. 73.

APPENDIX A

STUDY OF THE ORDER OF MAGNITUDE OF TERMS

A.1 A Study of Fluid Flow Equations:

Oscillatory flow of slightly compressible liquid in long, elastic, thick walled tubes with small internal diameters, is considered for the study of the order of magnitude of terms. The following assumptions can be made for this type of system.

$$\frac{R_i}{L} \ll 1$$

$$\frac{v_o}{c} \ll 1$$

$$\frac{R_i}{c} \ll 1$$

$$\frac{R_i}{c} \cong \frac{R_i}{L} \cong \frac{v_o}{c} \cong \epsilon \ll 1 \qquad \text{(A.1)}$$

For an elastic soft pipe line system the flow conditions can be given as follows.

v_o (the flow velocity) $<$ 3.0 ft/sec.

L (length of the tube) $\approx$ 10 ft

R_i (the inside radius) $\approx$ 0.01 ft

ω_n (the resonant frequency of oscillations in x direction) $\approx 1.3 \times 2\pi$ rad/sec.

c (the phase velocity in liquids) $\approx$ 100 ft/sec.

From these values, the order magnitude of the following quantities can be approximately estimated.

$$\frac{R_i}{L} \approx \frac{0.01}{10} = .001$$

$$\frac{v_o}{c} \approx \frac{3}{100} = .03$$

$$\frac{R_i}{c} \approx \frac{1.3 \times 2\pi \times 0.01}{100} \approx 0.001$$

The following nondimensional variables are assumed for nondimensionalising the continuity and momentum equations

$$\begin{array}{llll}
v = v_o v', & (v')_{max} \overset{O}{=} 1, & (v')_{avg} \overset{O}{=} 0, \\
u = \epsilon v_o u', & (u')_{max} \overset{O}{=} 1, & (u')_{avg} \overset{O}{=} 0, \\
p = \rho_o c v_o p', & (p')_{max} \overset{O}{=} 1, & (p')_{avg} \overset{O}{=} 0, \\
 = \rho_o \rho', & & \\
x = L\, x', & (x')_{max} \overset{O}{=} 1, & \\
r = R_i r', & (r')_{max} \overset{O}{=} 1, & \\
t = \frac{1}{\omega} t', & &
\end{array}$$

t' varies continuously, but the duration of each cycle is of the order $1/\omega$, that is of period $1/2\,\pi\,\omega$ (A.2)

Where the primes indicate nondimensional quantities and the subscript 0 indicates a characteristic constant value. In tubes of small internal diameter, the assumptions that the radial velocity component is of very small order of magnitude as compared to the axial flow velocity is justified, particularly so in rigid tubes. The nondimensional form of the pressure perturbations is assumed on the basis

of Joukowsky formula for pressure surges in wave transmission, where the pressure surge Δp is given by

$$\Delta p = \rho c \Delta v$$

for a velocity disturbance of Δv.

From Equation (2.1) continuity equation is given by

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial x} + \frac{u}{\rho} \frac{\partial \rho}{\partial r} + \frac{v}{\rho} \frac{\partial \rho}{\partial x} = 0 \qquad (A.3)$$

Now substituting the relationships in the set of Equations (A.2) in Equation (A.3) and by multiplying the resulting equation throughout by L/v_o yields the following equation

$$\frac{\omega L}{c} \frac{\partial \rho'}{\partial t'} + \frac{L\epsilon}{R_i} \left[\frac{\partial u'}{\partial r'} + \frac{u'}{r'} \right] + \frac{\partial v'}{\partial x'} + \frac{v_o}{c} \frac{v'}{\rho'} \frac{\partial \rho'}{\partial x'} +$$

$$\frac{v_o}{c} \frac{L}{R_i} \epsilon \frac{u'}{\rho'} \frac{\partial \rho'}{\partial r'} = 0 \qquad (A.4)$$

By considering the terms of the order of unity and neglecting the terms of the order of ϵ the continuity equation in this form with the corresponding terms in original form can be written as

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \qquad (A.5)$$

From Equation (2.2) the momentum equation in the axial direction is given by

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left| \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right| = 0$$

(A.6)

Substituting the relations in (A.2) and by multiplying the resulting equation throughout by $L/v_o c$, yields the following equation.

$$\frac{\omega L}{c}\frac{\partial v'}{\partial t'} + \frac{v_o}{c} v' \frac{\partial v'}{\partial x'} + \epsilon \frac{L}{R_i}\frac{v_o}{c} u' \frac{\partial v'}{\partial r'} + \frac{1}{\rho'}\frac{\partial p'}{\partial x'} -$$

$$\nu\left[\frac{1}{Lc}\frac{\partial^2 v'}{\partial x'^2} + \frac{L}{cR_i^{\,2}}\frac{\partial^2 v'}{\partial r'^2} + \frac{L}{cR_i}\frac{1}{r'}\frac{\partial v'}{\partial r'}\right] = 0 \quad (A.7)$$

Since the quantities R_i/L and v_o/c are of the order of ϵ in magnitude and hence, for inertia forces balanced by viscous and pressure forces

$$\frac{\nu}{cR_i} \simeq \epsilon$$

By neglecting the terms of the order of ϵ the momentum equation in the axial direction in this form with the corresponding terms in original form can be written as

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} - \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right] = 0 \qquad (A.8)$$

From Equation (2.3) the momentum equation in the radial direction is given by

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} +\right.$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right] = 0 \qquad (A.9)$$

Substituting the relations in (A.2) in Equation (A.9) and multiplying the resulting equation throughout by $L/v_o c$, the following equation is obtained

$$\frac{\epsilon\omega L}{c}\frac{\partial u'}{\partial t'} + \frac{v_o\epsilon}{c}\, v' \frac{\partial u'}{\partial x'} + \epsilon^2 \frac{v_o}{c}\frac{L}{R_i}\, u' \frac{\partial u'}{\partial r'} + \frac{L}{R_i}\frac{1}{\rho'}\frac{\partial p'}{\partial r'}$$

$$-\nu\left[\frac{\epsilon L}{cR_i^{\,2}}\frac{\partial^2 u'}{\partial r'^2} + \frac{\epsilon L}{cR_i^{\,2}}\frac{1}{r'}\frac{\partial u'}{\partial r'} - \frac{L\epsilon}{cR_i^{\,2}}\frac{u'}{r'^2} + \frac{\epsilon}{cL}\frac{\partial^2 u'}{\partial x'^2}\right] = 0$$

(A.10)

A comparison of Equations (A.7) and (A.10) shows that the terms in the radial momentum equation are of small order of magnitude, of order ϵ or smaller in magnitude, and hence, in the case of a laminar fluid flow in a very rigid pipe line, the radial momentum equation can be neglected. From Equation (A.10) $1/\rho^{*}\ \partial p/\partial r$ is of the order ϵ.

In case of the fluid flow in very soft tubes, where the radial velocity and radial momentums may be of considerable magnitude, the simplified radial momentum equation can be obtained by considering the terms of the order of magnitude ϵ. The radial momentum equation in this form with the corresponding terms written in original form is given as

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2}\right] = 0 \qquad \text{(A.11)}$$

The radial pressure gradient is of the order of magnitude ϵ and much smaller at still low frequencies. The radial oscillations are governed by a wave velocity which is determined by the fluid compressibility and density. These facts justify the assumption that the radial pressure gradient can be neglected. With this assumption, the

simplified radial momentum equation for liquid flows in an elastically soft tube can be written as

$$\frac{\partial u}{\partial t} - \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} \right] = 0 \qquad (A.12)$$

The simplified form of the continuity equation (A.5) is again considered for further simplification. In the case of the liquid flow in a long pipe line with comparatively small inside diameter, if the flow is considered as one dimensional, the average axial flow can be considered for the analysis of the dynamic flow. By taking a cross sectional average of Equation (A.5), which can be done by multiplying this equation throughout by $2\pi r dr$, integrating between the limits 0 and R_i and dividing by the cross sectional area πR_i^2, Equation (A.5) becomes

$$\frac{2\pi}{\pi R_i^2} \int_0^{R_i(x,t)} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} + \frac{u}{r} \right] r\, d\, r = - \frac{2\pi}{\pi R_i^2} \cdot$$

$$\int_0^{R_i(x,t)} \frac{1}{\rho} \frac{\partial p}{\partial t}\, r\, d\, r - \frac{1}{\pi R_i^2} \frac{\partial (\pi R_i^2)}{\partial t} \qquad (A.13)$$

If the values of the average flow velocity and the average density are defined by the equation

$$\bar{v} = \frac{2\pi}{\pi R_i^2} \int_0^{R_i} v\, r\, d\, r$$

$$\bar{\rho} = \frac{2\pi}{\pi R_i^2} \int_0^{R_i} \rho\, r\, d\, r$$

Equation (A.13) can be written as

$$\frac{\partial \bar{v}}{\partial x} - \frac{v(R_i,x,t)}{R_i}\frac{\partial R_i}{\partial x} + \frac{2\pi}{\pi R_i^2}\int_0^{R_i(x,t)} \left| \frac{\partial u}{\partial r} + \frac{u}{r} \right| r\, d\, r$$

$$= - \frac{1}{\pi R_i^2}\frac{\partial(\pi R_i^2)}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{\rho}\frac{\rho(R_i,x,t)}{R_i}\frac{\partial R_i}{\partial t}$$

(A.14)

where $\int_0^{R_i}\frac{1}{\rho}\frac{\partial}{\partial t}\, r\, d\, r = \int_0^{R_i}\frac{1}{\bar{\rho}}\frac{\partial}{\partial t}\quad r\, d\, r$

Now $\int_0^{R_i}\left[\frac{\partial u}{\partial r} + \frac{u}{r}\right] r\, d\, r = \int_0^{R_i} r\frac{\partial u}{\partial r}\, dr + u\,(R_i',x,t)\; R_i$

$$- \int_0^{R_i} r\frac{\partial u}{\partial r}\; d\, r$$

$$= u(R_i,x,\; t)\; R_i \qquad (A.15)$$

By substituting Equation (A.15) into Equation (A.14) and by assuming

$$\frac{1}{v_o}\;\frac{\partial R_i}{\partial t} \;\underline{\circ}\; \epsilon^2$$

$$\frac{\partial R_i}{\partial x} \;\underline{\circ}\; \epsilon^2$$

$$\frac{u(R_i)}{v_o} \;\underline{\circ}\; \epsilon^2 \qquad (A.16)$$

Equation (A.14) can be simplified to the form

$$\frac{\partial \bar{v}}{\partial x} + \frac{1}{\rho}\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{\pi R_i^2}\frac{\partial(\pi R_i^2)}{\partial t} = 0 \qquad (A.17)$$

Equation (A.17) can also be written in the form

$$\frac{\partial \bar{v}}{\partial x} + \frac{1}{B}\frac{\partial p}{\partial t} = 0 \qquad (A.18)$$

where

$$\frac{1}{B} = \frac{1}{\bar{\rho}}\left[\frac{\partial \bar{\rho}}{\partial p}\right]_s + \frac{1}{\bar{\rho}\pi R_i^2}\left[\frac{\partial(\bar{\rho}\pi R_i^{\,2})}{\partial p}\right]_s$$

The assumption that the radial velocity and the variations in inside radius are small, is partly justified from the calculation of these quantities.

A.2 A Study of the Equations for the Elastic Tubes:

Tube momentum equation for the longitudinal and radial oscillation are given by

$$\rho' \frac{\partial^2 z}{\partial t^2} = (\lambda + 2G) \frac{\partial^2 z}{\partial x^2} + (\lambda + G) \left[\frac{\partial^2 y}{\partial x \partial z} + \frac{1}{r} \frac{\partial y}{\partial z} \right] + G \left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} \right] \qquad (A.19)$$

$$\rho' \frac{\partial^2 y}{\partial t^2} = (\lambda + 2G) \left[\frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} - \frac{y}{r^2} \right] + (\lambda + G) \frac{\partial^2 z}{\partial r \partial x} + G \frac{\partial^2 y}{\partial x^2} \qquad (A.20)$$

The solutions for the oscillatory motion of the tube wall can be assumed as

$$z = D_1 \, e^{i\alpha(x - c_1 t)} \qquad (A.21a)$$

$$y = \frac{D_2}{r} \, e^{i\alpha(x - c_2 t)} \qquad (A\text{-}21b)$$

where D_1 and D_2 are constants,

c_1 is the longitudinal wave transmission velocity

c_2 is the transverse wave transmission velocity

Substituting Equations (A.21) into Equations (A.19) and (A.20) and separating the oscillatory components, yields the following equations

$$\rho' D_1 \left(-\alpha^2 c_1^2 \right) = (\lambda + 2G) D_1 (-\alpha^2) \qquad (A.22)$$

$$\rho' \frac{D_2}{r} (-\alpha^2 c_2^2) = G \frac{D_2}{r} (-\alpha^2) \qquad (A.23)$$

From Equations (A.22) and (A.23)

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho'}} \quad ;$$

$$c_2 = \sqrt{\frac{G}{\rho'}}$$

Equations (A.22) and (A.23) with the terms written in the original form without damping can be written as

$$\rho' \frac{\partial^2 z}{\partial t^2} = (\lambda + 2G) \frac{\partial^2 z}{\partial x^2} \qquad (A.24)$$

$$\rho' \frac{\partial^2 y}{\partial t^2} = G \frac{\partial^2 y}{\partial x^2} \qquad (A.25)$$

APPENDIX B

SOME DETAILED STEPS OF CHAPTER IV

B.1 Details of the Steps Between Equations (4.19) and (4.20):

Equations (4.19) and (4.18) are repeated below

$$z_x = \int_0^L z_{xx_o}\, dx_o \tag{B.1}$$

$$z_{xx_o} = \frac{2C_1\, h\,(x_o,\ j\omega)}{L}\, \Sigma\ \frac{\sin\frac{n\pi x}{L}\ \sin\frac{n\pi x_o}{L}}{(-\rho'\omega^2 + \alpha^2 n^2 \pi^2/L^2)} \tag{B.2}$$

Substituting Equation (B.2) in Equation (B.1) yields

$$z_x = \frac{2C_1}{L}\, \Sigma\ \frac{\sin\frac{n\pi x}{L}}{(-\rho'\omega^2 + \alpha^2 n^2 \pi^2/L^2)} \int_0^L h(x_o,\ j\omega) \sin\frac{n\pi x_o}{L}\ dx_o \tag{B.3}$$

From Equation (3.20) $h(x_o,\ j\omega)$ is given by

$$h(x_o,\ j\omega) = G_1 \cosh\left(\gamma x_o\right) + H_1 \sinh\left(\gamma x_o\right) \tag{B.4}$$

$$\int_0^L h(x_o, j\omega)\, \sin\frac{n\pi x_o}{L}\, dx_o = G_1 \int_0^L \cosh\left(\gamma x_o\right) \sin\frac{n\pi x_o}{L}\, dx_o$$
$$+ H_1 \int_0^L \sinh\left(\gamma x_o\right)\, \sin\frac{n\pi x_o}{L}\, dx_o \tag{B.5}$$

The following integrals are taken from the standard integral tables.

$$\int_0^L \cosh(\gamma x_o) \sin\frac{n\pi x_o}{L}\, dx_o = \left[\frac{\gamma \sinh\left(\gamma x_o\right) \sin\frac{n\pi x_o}{L} - \frac{n\pi}{L} \cosh\left(\gamma x_o\right) \cos\frac{n\pi x}{L}}{\gamma^2 + n^2\pi^2/L^2}\right. \tag{B.6}$$

$$\int_0^L \sinh(\gamma x_o) \sin \frac{n\pi x_o}{L}\, dx_o = \left[\frac{\gamma \cosh(\gamma x_o) \sin \frac{n\pi x_o}{L} - \frac{n\pi}{L} \sinh(\gamma x_o) \cos \frac{n\pi x_o}{L}}{(\gamma^2 + n^2\pi^2/L^2)} \right]$$

(B.7)

Now by substituting Equations (B.6) and (B.7) in Equation (B.5) and taking limits gives the following equation.

$$\int_0^L h(x_o, j\omega) \sin \frac{n\pi x_o}{L}\, dx_o = \frac{n\pi}{L} \left[\frac{G_1(1-(-1)^n \cosh(\gamma L)) - H_1(-1)^n \sinh(\gamma L)}{\gamma^2 + n^2\pi^2/L^2} \right]$$

(B.8)

Substituting Equation (B.8) in Equation (B.3) yields

$$z_x = C_2 \Sigma b_n \sin \frac{n\pi x}{L} \tag{B.9}$$

where

$$C_2 = \frac{2C_1\pi}{L} \quad ; b_n = \frac{n[G_1(1-(-1)^n \cosh(\gamma L)) - H_1(-1)^n \sinh(\gamma L)]}{(-\rho'\omega^2 + \alpha^2 n^2\pi^2/L^2)(\gamma^2 + n^2\pi^2/L^2)}$$

(B.10)

B.2 <u>Details of the Steps Between Equations (4.29) and (4.30):</u>

Equations (4.28), (4.26) and (4.29) are repeated below

$$\bar{v}_x' = K_2\ g(x,\ j\omega) + j\omega z_x \qquad (B.11)$$

$$\frac{dp_x'}{dx} = j\,\rho\omega\ \left[\ g\ (x,\ j\omega)\ I_o\ (K\ R_i) - j\omega z_x\ \right] \qquad (B.12)$$

$$\frac{dP_x}{dx} = \frac{jB}{\omega}\ \frac{d\bar{v}_x'}{dx} \qquad (B.13)$$

Substituting Equations (B.11) and (B.12) in Equation (B.13) yields the following equation

$$j\rho\omega \left|\, g(x,\ j\omega)\ I_o(K\ R_i) - j\omega z_x\ \right| = \frac{jB}{\omega}\ \left|\ K_2\ \frac{d^2 g(x, j\omega)}{dx^2} + \frac{j\omega d^2 z_x}{dx^2}\ \right| \qquad (B.14)$$

Now substituting Equation (B.9) in Equation (B.14) and rearranging the terms yield

$$\frac{g(x, j\omega)\ I_o(K\ R_i)\,\omega^2}{K_2\ c_o^{\ 2}} - \frac{\omega^2}{c_o^{\ 2}\ K_2}\ j\omega C_2\ \Sigma\ b_n\ \ \sin\frac{n\pi x}{L}$$

$$= \frac{d^2\ g(x, j\omega)}{dx^2} - j\omega C_2\ \ \Sigma\ b_n\ \frac{n^2\pi^2}{L^2}\ \sin\frac{n\pi x}{L}$$

Further rearrangement yields

$$\frac{d^2 g(x, j\omega)}{dx^2} - \gamma^2\ g(x, j\omega) = \frac{j\omega C_2}{K_2}\ \Sigma\ \left(\frac{n^2\pi^2}{L^2} - \frac{\omega^2}{c_o^{\ 2}}\right)\ b_n\ \sin\frac{n\pi x}{L} \qquad (B.15)$$

where $\gamma^2 = \dfrac{\beta^2\,\omega^2}{c_o^{\ 2}}$; $\beta^2 = \dfrac{I_o(K\ R_i)}{K_2}$

APPENDIX C

ESTIMATION OF THE EFFECTIVE BULK MODULUS OF LIQUIDS IN ELASTIC LINES

Estimation of the Effective Bulk Modulus of Liquids in Elastic Lines

Due to the radial expansion of the tube caused by the internal pressure, the volumetric capacity of a given length of the pipe line is increased. Hence the volume of the fluid that has to be introduced into a given length of the tube to raise the internal pressure is the sum of the volumetric deformation of the fluid due to its compressibility and the internal volumetric deformation of the pipe line due to internal pressure.

Considering a pipe line without end restrictions as a thick cylinder with inside radius R_i, outside radius R_o and the Poisson's ratio of the material σ, the radial expansion of the inside radius R_i for a pressure variation Δp is given by the equation[16]

$$\Delta R_i = \frac{\Delta p}{E} \frac{R_i^3}{(R_o^2 - R_i^2)} \left[(1-\sigma) + (1+\sigma) R_o^2/R_i^2\right]$$

$$= {}^{m}1/E\ (\Delta p) \qquad (C.1)$$

Considering a unit length of the tube, the change in fluid volume per unit length caused by the pressure variation of Δp is given by

$$\Delta V = \pi \left[(R_i + \Delta R_i)^2 - R_i^2\right]$$

or $$\Delta V = \pi\ (2R_i + \Delta R_i)\ \Delta R_i \qquad (C.2)$$

For small pressure variations, ΔR_i is neglected compared with 2 R_i and hence the variation of the fluid volume is given by

$$\Delta V = 2 \pi R_i \, \Delta R_i \qquad (C.3)$$

Substituting for ΔR_i from Equation (C.1) Equation (C.3) reduces to the form

$$\Delta V = 2 \pi R_i \frac{m_1}{E} \Delta p \qquad (C.4)$$

From the equation of state (2.4), the volumetric strain in the liquid due to small pressure variations can be written as

$$\frac{\Delta V_a}{V_a} = - \frac{\Delta \rho}{\rho} = - \frac{\Delta p}{B'} \qquad (C.5)$$

where B' is the isentropic liquid bulk modulus

Hence the total volumetric strain of the liquid and pipe line system, which is also equal to the volume of the fluid that has to be introduced into a unit internal volume of the pipe line system, can be obtained as

$$\frac{\Delta V_t}{V_t} = - \frac{\Delta p}{B'} + \frac{\Delta V}{V}$$

or $$\frac{\Delta V_t}{V_t} = - \frac{\Delta p}{B'} + \frac{2\pi R_i m_1}{E} \frac{\Delta p}{\pi R_i^2}$$

or $$\frac{\Delta V_t}{V_t} = - \Delta p \left[\frac{1}{B'} + \frac{2m_1}{ER_i} \right] \qquad (C.6)$$

The effective bulk modulus B of the liquid can be

obtained as

$$\frac{\Delta V_t}{V_t} = - \frac{\Delta p}{B} = - \Delta p \left[\frac{1}{B'} + \frac{2m_1}{ER_i} \right]$$

or

$$\frac{1}{B} = \frac{1}{B'} + \frac{2m_1}{ER_i} \tag{C.7}$$

This equation can be used to obtain the effective bulk modulus when the variations in the inside radius are considered to be quite small compared to the inside radius.

APPENDIX D

DETAILS OF THE STEPS BETWEEN EQUATIONS (5.15) AND (5.16)

Details of the Steps Between Equations (5.15) and (5.16)

Equations (5.14) and (5.15) are repeated below

$$y_{xx_o} = C_3 \, P_{x_o} \, \Sigma \frac{\sin n\pi x/L \, \sin n\pi x_o/L}{(-\rho'\omega^2 + \alpha_T{}^2 n^2\pi^2/L^2)} \tag{D.1}$$

$$y_x = \int_0^L y_{xx_o} \, dx_o \tag{D.2}$$

By substituting (D.1) in (D.2) yields

$$y_x = C_3 \, \Sigma \frac{\sin \; n\pi x/L}{(-\rho'\omega^2 + \alpha_T{}^2 \, n^2\pi^2/L^2)} \int_0^L P_{x_o} \, \sin \frac{n\pi x_o}{L} \, dx_o \tag{D.3}$$

From Equation (3.23), P_{x_o} is given by

$$P_{x_o} = C_8 \left[G_L \sinh(\gamma x_o) + H_1 \cosh(\gamma x_o) \right]$$

Then

$$\int_0^L P_{x_o} \sin \frac{n\pi x_o}{L} \, dx_o = C_8 \left[G_1 \int_0^L \sinh(\gamma x_o) \sin \frac{n\pi x_o}{L} \, dx_o + H_1 \int_0^L \cosh(\gamma x_o) \sin \frac{n\pi x_o}{L} \, dx_o \right] \tag{D.4}$$

From the integrals given in (B.6) and (B.7), Equation (D.4) reduces to the form

$$\int_0^L P_{x_o} \sin \frac{n\pi x_o}{L} \, dx_o = C_8 \frac{n\pi}{L} \left[\frac{G_1(-1)^n \sinh(\gamma L) + H_1(1-(-1)^n \cosh(\gamma}{{}^2 + n^2\pi^2/L^2} \right. \tag{D.5}$$

Substituting Equation (D.5) in Equation (D.3) yields

$$y_x = C_4 \Sigma b_n \sin \frac{n\pi x}{L} \tag{D.6}$$

where $C_4 = \dfrac{C_8 \pi C_3}{\beta L}$

$$b_n = n \left[\frac{-G_1(-1)^n \sinh(\gamma L) + H_1(1-(-1)^n \cosh(\gamma L))}{(-\rho'\omega^2 + \alpha_T^2 n^2\pi^2/L^2)(\gamma^2 + n^2\pi^2/L^2)} \right]$$

APPENDIX E

LOAD IMPEDENCE ANALYSIS FOR A MERCURY COLUMN IN A VERTICAL U-TUBE

Load Impedence Analysis for a Mercury Column in a Vertical U-Tube

The dynamic response of the fluid is very much influenced by the end conditions of the pipe line. Usually the inlet end is without any restrictions and it is connected to a supply source. There are several possibilities of loading the output or load end of the pipe line such as a closed end, an open end, the end with a flow restriction and an end connected to load system with characteristics such as inertia, damping, capacitance due to compressibility of the liquid and Čoulomb friction. Load end conditions can easily be defined by introducing the term load impedence, which is the ratio of the instantaneous pressure to the instantaneous velocity of the fluid at the output end of the pipe line. The term load admittance is the inverse of the load impedence. For an open exit the load impedence is zero and load admittance is infinite, because the gauge pressure at the exit is zero. For a closed exit the load impedence is infinite and load admittance is zero, because the instantaneous velocity component at the output end is zero. Srinivas[14] has given a method of estimating the load impedence for a load system with mercury column oscillating in a vertical U-tube and is given as follows.

Neglecting the damping effect of mercury in a glass tube, the force acting on the mercury column and the load pressure, inertia of the column of mercury and the restoring force due to the gravitational field. Hence the force

balance equation can be written as

$$P_L V_L = m_L \dot{w}_L + K_S A_L \int w_L \, dt \tag{E.1}$$

For frequency response analysis the time derivative and the time integration in Equation (E.1) is replaced by $j\omega$ and $1/j\omega$. By the above substitution Equation (E.1) reduces to the form

$$P_L = \frac{1}{A_L} \left[j\omega m_L + \frac{K_S A_L}{j\omega} \right] W_L \tag{E.2}$$

The ratio of the velocity of the liquid at the pipe line and at the load is given by

$$\frac{W_L}{V_L} = \frac{A_p}{A_L}$$

and hence the load impedence of this system is obtained as

$$\frac{P_L}{V_L} = j \left[m_L \frac{A_p}{A_L^2} \omega - \frac{K_S A_p}{\omega A_L} \right] \tag{E.3}$$